BETTI NUMBERS OF ORDER PRESERVING GRAPH HOMOMORPHISMS

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ABSTRACT. For graphs G and H with totally ordered vertex sets, a function mapping the vertex set of G to the vertex set of H is an order preserving homomorphism from G to H if it is nondecreasing on the vertex set of G and maps edges of G to edges of H. In this paper, we study order preserving homomorphisms whose target graph H is the complete graph on n vertices. By studying a family of graphs called nonnesting arc diagrams, we are able to count the number of order preserving homomorphisms (and more generally the number of order preserving multihomomorphisms) mapping any fixed graph G to the complete graph K_n.

CONTENTS

1. INTRODUCTION

The study of graph homomorphisms has been the subject of a great deal of recent work in the fields of enumerative, algebraic, and topological combinatorics. A recent survey of Lovász et. al. [1] is an excellent source on the many facets of enumerating graph homomorphisms, while Kozlov's mongraph [3] outlines a more topological approach. In this paper, we study combinatorial properties of order preserving homomorphisms between two graphs G and H as introduced by Braun, Browder and Klee [2].

Throughout this paper, $V(G)$ and $E(G)$ will denote the vertex set and edge set respectively of a graph G. All graphs are assumed to be simple, meaning that loops and multiple edges are not allowed.

Let G be a graph on vertex set $[m] = \{1, 2, \ldots, m\}$ and let H be a graph on vertex set $\{x_1, x_2, \ldots, x_n\}$. We order the vertex set of G naturally, and we order the vertex set of H by

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declaring that $x_1 < x_2 < \cdots < x_n$. An *order preserving homomorphism* from G to H is a function $\varphi : V(G) \to V(H)$ such that

(1) If $1 \leq i < j \leq m$, then $\varphi(i) \leq \varphi(j)$, and
(2) If $(i,j) \in E(G)$, then $(\varphi(i), \varphi(j)) \in E(H)$.

An order preserving homomorphism $\varphi : G \to H$ may be presented as a vector $[\varphi(i)]_{i=1}^m = [\varphi(1), \ldots, \varphi(m)]$.

Example 1.1. Let G and H be the graphs shown in Figure 1.1.

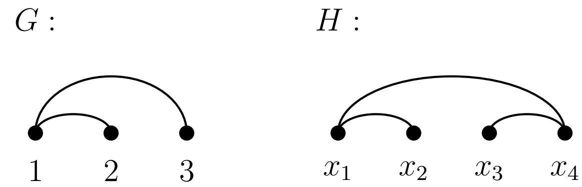

FIGURE 1

Define functions $\varphi_1, \varphi_2, \varphi_3 : V(G) \to V(H)$ as follows:

$$\begin{aligned} \varphi_1 : &\qquad [x_1, x_2, x_2] \\ \varphi_2 : &\qquad [x_1, x_2, x_4] \\ \varphi_3 : &\qquad [x_1, x_3, x_4]. \end{aligned}$$

The functions φ_1 and φ_2 are order preserving homomorphisms from G to H. Notice that since $(2,3)$ is not an edge in G, having $\varphi_1(2) = \varphi_1(3)$ does not violate the definition of an order preserving homomorphism. The function φ_3 is order preserving, but it is not a homomorphism since $(1,2) \in E(G)$, but $(\varphi(1), \varphi(2)) = (x_1, x_3) \notin E(H)$.

Rather than view each order preserving homomorphism from G to H as a single function, it is often more convenient to encode several homomorphisms as a single object. An *(order preserving) multihomomorphism* from G to H is a function $\eta : V(G) \to 2^{V(H)} \backslash \emptyset$ with the property that $[\varphi(i)]_{i=1}^m$ is an order preserving homomorphism from G to H for all possible choices of $\varphi(i) \in \eta(i)$ and $1 \leq i \leq m$. The *complex of order preserving homomorphisms* from G to H, denoted $\mathrm{OHOM}(G,H)$, is the collection of all multihomomorphisms from G to H.

For any graphs G and H, there is a geometric cell complex corresponding to $\mathrm{OHOM}(G,H)$ whose faces are labeled by multihomomorphisms from G to H. While the geometry of $\mathrm{OHOM}(G,H)$ is very interesting in its own right, it is not the primary focus of this paper, and we will not spend any further time discussing it. For reasons that are motivated by this underlying geometry, we define the *dimension* of a multihomomorphism $\eta \in \mathrm{OHOM}(G,H)$ to be

$$\dim(\eta) := \sum_{i=1}^{m} (|\eta(i)| - 1).$$

A zero-dimensional multihomomorphism is an order preserving homomorphism. In this paper, we are primarily interested in a family of combinatorial invariants of $\mathrm{OHOM}(G,H)$ called its Betti numbers.

Definition 1.2. The *r-th Betti number* of the complex $\mathrm{OHOM}(G,H)$, denoted $\beta_r(G,H)$, counts the number of multihomomorphisms $\eta \in \mathrm{OHOM}(G,H)$ with $\dim(\eta) = r$.

Example 1.3. Let G and H be as in Example 1.1. The following table encodes a one-dimensional multihomomorphism $\eta \in \mathrm{OHOM}(G,H)$. The two distinct choices of elements $[\varphi(1),\varphi(2),\varphi(3)]$ correspond to the order preserving homomorphisms φ_1 and φ_2 of Example 1.1.

$\eta(1)$	$\eta(2)$	$\eta(3)$
x_1	x_2	x_2
		x_4

TABLE 1. A multihomomorphism $\eta \in \mathrm{OHOM}(G,H)$

The following proposition is a consequence of from our definitions of order preserving homomorphisms. We introduce the following notation, which will be used for the remainder of the paper. If X and Y are subsets of some totally ordered set (for our purposes, either $[m]$ or $\{x_1,\dots,x_n\}$), we write $X \le Y$ (or $X < Y$) to indicate that $x \le y$ (similarly $x < y$) for all $x \in X$ and all $y \in Y$.

Proposition 1.4. *Let G and H be graphs with $V(G) = [m]$ and $V(H) = \{x_1,\dots,x_n\}$. If $\eta \in \mathrm{OHOM}(G,H)$, then $\eta(1) \le \eta(2) \le \cdots \le \eta(m)$. Moreover, if (i,j) is an edge in G, then $\eta(i) < \eta(j)$.*

The purpose of this paper is to determine the Betti numbers $\beta_r(G,K_n)$ of the complex of order preserving homomorphisms between a fixed graph G and the complete graph on n vertices. In order to more easily compute the Betti numbers $\beta_r(G,K_n)$, we use the following series of reductions outlined in [2, Section 5]. All relevant definitions are deferred to Section 2.

(1) We show that for any graph G, there is a nonnesting partition $\mathcal{P}$ of $[m]$ and a corresponding graph $\Gamma_{\mathcal{P}}$ on $[m]$, called an *arc diagram*, such that $\mathrm{OHOM}(G,K_n) = \mathrm{OHOM}(\Gamma_{\mathcal{P}},K_n)$.
(2) We define a weight function $\omega_r(\Gamma_{\mathcal{P}},K_n)$ that counts the number of r-dimensional multihomomorphisms in $\mathrm{OHOM}(\Gamma_{\mathcal{P}},K_n)$ that are "minimally" determined by $\mathcal{P}$. These weights are ultimately easier to compute than the Betti numbers of $\mathrm{OHOM}(\Gamma_{\mathcal{P}},K_n)$.
(3) We define a partial order, denoted $\preceq$, on the family of nonnesting partitions of $[m]$ and show that
$$\beta_r(\Gamma_{\mathcal{P}},K_n) = \sum_{\mathcal{Q}\preceq\mathcal{P}} \omega_r(\Gamma_{\mathcal{Q}},K_n).$$

In Section 3, we provide an explicit (and simple) closed formula for the weight function $\omega_r(\Gamma_{\mathcal{P}},K_n)$ for any nonnesting partition $\mathcal{P}$.

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2. Nonnesting partition graphs

2.1. **Nonnesting partitions.** A *partition* $\mathcal{P} = \{P_1, \dots, P_t\}$ of the set $[m]$ is a collection of nonempty subsets $P_i \subseteq [m]$ (called *blocks*) such that $P_i \cap P_j = \emptyset$ for all $i \neq j$ and $P_1 \cup \cdots \cup P_t = [m]$. We say that two blocks P_i and P_j *nest* if there exist $1 \leq a < b < c < d \leq m$ with $\{a, d\} \subseteq P_i$ and $\{b, c\} \subseteq P_j$ and there does not exist $e \in P_i$ with $b < e < c$. If no pair of blocks of $\mathcal{P}$ nest, we say that $\mathcal{P}$ is a *nonnesting partition* of $[m]$. The family of nonnesting partitions was originally introduced and studied by Postnikov [4, Remark 2].

Example 2.1. The partition $\mathcal{P}_1 = \{\{1,4\},\{2,5,6\},\{3\}\}$ of $[6]$ is a nonnesting partition. The partition $\mathcal{P}_2 = \{\{1,3,5\},\{2,6\},\{4\}\}$ is nesting since the blocks $\{1,3,5\}$ and $\{2,6\}$ nest.

It is more illuminating to represent a partition $\mathcal{P}$ of $[m]$ as a graph $\Gamma_{\mathcal{P}}$ as follows.

Definition 2.2. Let $\mathcal{P}$ be a partition of $[m]$ and let $P_i = \{i_1, \dots, i_k\}$ be a block of $\mathcal{P}$ with $i_1 < \cdots < i_k$. The *arc diagram* $\Gamma_{\mathcal{P}}$ is the graph on vertex set $[m]$ whose edges are given by (i_j, i_{j+1}) for consecutive elements of P_i taken over all blocks of $\mathcal{P}$.

The name "arc diagram" is natural when the graph $\Gamma_{\mathcal{P}}$ is drawn so that its vertices are placed in a line and its edges are drawn as upper semicircular arcs, as shown in Example 2.3. In this representation, a partition $\mathcal{P}$ is nonnesting exactly when no arc of $\Gamma_{\mathcal{P}}$ is nested below another.

Example 2.3. Let $\mathcal{P}_1 = \{\{1,4\},\{2,5,6\},\{3\}\}$ and $\mathcal{P}_2 = \{\{1,3,5\},\{2,6\},\{4\}\}$ be the partitions of $[6]$ discussed in Example 2.1. The arc diagrams $\Gamma_{\mathcal{P}_1}$ and $\Gamma_{\mathcal{P}_2}$ are shown in Figures 2 and 3.

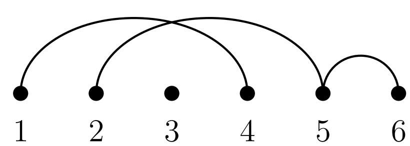

Figure 2. The arc diagram for $\{\{1,4\},\{2,5,6\},\{3\}\}$.

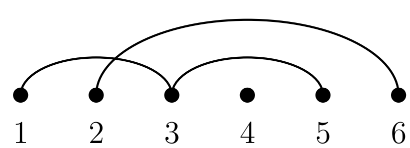

Figure 3. The arc diagram for $\{\{1,3,5\},\{2,6\},\{4\}\}$.

The following proposition shows that in order to compute Betti numbers $\beta_r(G, K_n)$ for arbitrary graphs G, we need only study the Betti numbers of nonnesting arc diagrams.

Proposition 2.4. [2, Proposition 5.6] *For any graph G on vertex set $[m]$, there exists a unique nonnesting partition $\mathcal{P}$ of $[m]$ such that $\Gamma_{\mathcal{P}}$ is a subgraph of G and* $\mathrm{OHOM}(G, K_n) = \mathrm{OHOM}(\Gamma_{\mathcal{P}}, K_n)$. *We call $\Gamma_{\mathcal{P}}$ the reduced arc diagram for G.*

Suppose there exist vertices $1 \leq a \leq b < c \leq d \leq m$ in G such that $(a, d), (b, c) \in E(G)$ (so that the edge (b, c) is nested below the edge (a, d)), and let G' be the graph obtained from G by removing the edge (a, d). The proof of Proposition 2.4 uses the observation that $\mathrm{OHOM}(G, K_n) = \mathrm{OHOM}(G', K_n)$ so that the reduced graph $\Gamma_{\mathcal{P}}$ is obtained from G by inductively removing the "top" arc in any pair of nested edges in G.

The goal for the remainder of this section is to describe a natural partial order on the family of nonnesting partitions of $[m]$. We then describe how to use this partial order to compute the Betti numbers $\beta_r(\Gamma_{\mathcal{P}}, K_n)$ of an arc diagram. For further information on posets and definitions of any undefined terms, we refer the reader to Stanley's book [5].

Definition 2.5. The *m-th diagram poset*, denoted $\mathcal{D}_m = (\mathcal{D}_m, \preceq)$, is the poset whose elements are arc diagrams of nonnesting partitions of $[m]$, partially ordered by $\mathcal{P} \preceq \mathcal{Q}$ if every arc of $\mathcal{Q}$ lies above an arc of $\mathcal{P}$.

The minimal element of $\mathcal{D}_m$ is the path of length $m-1$ on $[m]$, and the maximal element of $\mathcal{D}_m$ is the empty graph.

For example, there are five nonnesting partitions of [3]:

$$\begin{aligned}
\mathcal{P}_1 &= \{\{1\}, \{2\}, \{3\}\}, \\
\mathcal{P}_2 &= \{\{1, 3\}, \{2\}\}, \\
\mathcal{P}_3 &= \{\{1, 2\}, \{3\}\}, \\
\mathcal{P}_4 &= \{\{1\}, \{2, 3\}\}, \text{ and} \\
\mathcal{P}_5 &= \{\{1, 2, 3\}\}.
\end{aligned}$$

Let $\Gamma_1, \ldots, \Gamma_5$ denote their corresponding arc diagrams, as shown in Figure 4.

If $(P, \leq)$ is a poset, a subset $U \subseteq P$ is a *upper order ideal* if $y \in U$ whenever $x \in U$ and $y \geq x$. An upper order ideal $U \subseteq P$ is *principal* if there is an element $\alpha \in P$ such that $U = \{y \in P : y \geq \alpha\}$. The importance of the partial order on $\mathcal{D}_m$ is illustrated in the following proposition.

Proposition 2.6. [2, Proposition 5.8] *If $\mathcal{P} \preceq \mathcal{Q}$ in $\mathcal{D}_m$, then*

$$\mathrm{OHOM}(\Gamma_{\mathcal{P}}, K_n) \subseteq \mathrm{OHOM}(\Gamma_{\mathcal{Q}}, K_n).$$

Further, for each multihomomorphism $\eta \in \mathrm{OHOM}(G_e, K_n)$, where G_e denotes the empty graph on vertex set $[m]$, the upper order ideal $U(\eta) \subseteq \mathcal{D}_m$ of arc diagrams whose OHOM *complexes contain η is principal.*

Proof. Fix a multihomomorphism $\eta \in \mathrm{OHOM}(\Gamma_{\mathcal{P}}, K_n)$. We need to show that each choice $[\varphi(i) \in \eta(i)]_{i=1}^m$ yields an order preserving homomorphism from $\Gamma_{\mathcal{Q}}$ to K_n so that $\eta \in \mathrm{OHOM}(\Gamma_{\mathcal{Q}}, K_n)$ as well.

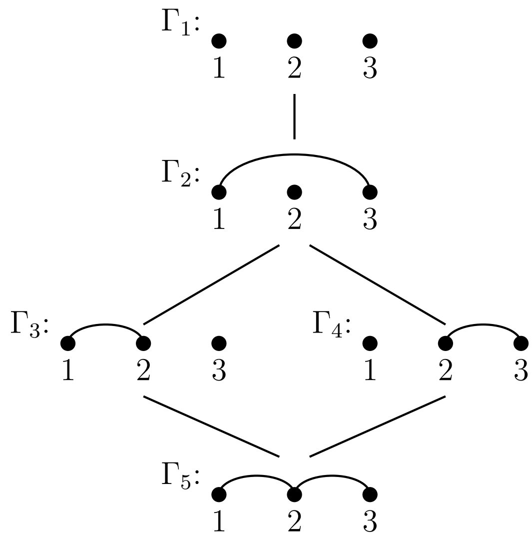

FIGURE 4. The Hasse diagram for $\mathcal{D}_3$.

Let (a,d) be an edge in $\Gamma_{\mathcal{Q}}$ with $a < d$. Since $\mathcal{P} \preceq \mathcal{Q}$, there is an edge (b,c) in $\Gamma_{\mathcal{P}}$ such that $a \leq b < c \leq d$. Since φ is an order preserving homomorphism from $\Gamma_{\mathcal{P}}$ to K_n and (b,c) is an arc in $\Gamma_{\mathcal{P}}$, we see that $\varphi(a) \leq \varphi(b) < \varphi(c) \leq \varphi(d)$. The arc (a,d) was arbitrary, and hence $\varphi(a) < \varphi(d)$ for all arcs (a,d) in $\Gamma_{\mathcal{Q}}$. Thus φ is an order preserving homomorphism from $\Gamma_{\mathcal{Q}}$ to K_n and $\eta \in \mathrm{OHOM}(\Gamma_{\mathcal{Q}}, K_n)$, as desired.

Suppose next that $\eta \in \mathrm{OHOM}(G_e, K_n)$. Consider the graph G on $[m]$ obtained as the union of all arc diagrams $\Gamma_{\mathcal{Q}}$ such that $\eta \in \mathrm{OHOM}(\Gamma_{\mathcal{Q}}, K_n)$, and let $\Gamma_{\mathcal{P}}$ denote the reduced arc diagram of G. Clearly $\mathcal{P} \preceq \mathcal{Q}$ for all nonnesting partitions $\mathcal{Q}$ whose OHOM complexes contain η. Thus $U(\eta)$ is generated by $\mathcal{P}$. $\square$

Example 2.7. We illustrate Proposition 2.6 for the following multihomomorphism $\eta \in \mathrm{OHOM}(\Gamma_1, K_9)$ using the notation from Figure 4.

$\eta(1)$	$\eta(2)$	$\eta(3)$
x_1	x_4	x_7
x_3	x_6	x_9
	x_7	

TABLE 2. A multihomomorphism $\eta \in \mathrm{OHOM}(\Gamma_1, K_9)$.

Since $\eta(2) \cap \eta(3) \neq \emptyset$, the nonnesting partitions $\mathcal{P}$ for which $\eta \in \mathrm{OHOM}(\Gamma_{\mathcal{P}}, K_9)$ are $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. The corresponding graphs Γ_1, Γ_2, and Γ_3 form an upper order ideal in $\mathcal{D}_3$ that is generated by Γ_3.

2.2. **Weights of nonnesting partition graphs.** Proposition 2.6 gives a well defined notion of the minimal arc diagram $\Gamma_{\mathcal{Q}}$ whose OHOM complex supports a given multihomorphism $\eta \in \mathrm{OHOM}(G_e, K_n)$. We make this more precise in the following definition.

Definition 2.8. Let $\mathcal{P}$ be a nonnesting partition of $[m]$. The *r-th weight* of $\mathcal{P}$ for n, denoted $\omega_r(\mathcal{P}, n)$, counts the number of r-dimensional multihomomorphisms $\eta \in \mathrm{OHOM}(G_e, K_n)$ such that $\mathcal{P}$ generates $U(\eta)$.

To be more specific, Proposition 2.6 says that for each nonnesting partition $\mathcal{Q}$ and each multihomomorphism $\eta \in \mathrm{OHOM}(\Gamma_{\mathcal{Q}}, K_n)$, there is a unique minimal nonnesting partition $\mathcal{P} \preceq \mathcal{Q}$ such that $\eta \in \mathrm{OHOM}(\Gamma_{\mathcal{P}}, K_n)$. This allows us to partition the r-dimensional multihomomorphisms of $\mathrm{OHOM}(\Gamma_{\mathcal{Q}}, K_n)$ according to the poset $\mathcal{D}_m$, as the following proposition indicates.

Proposition 2.9. [2, Proposition 5.10] *For any nonnesting partition $\mathcal{Q}$,*

$$\beta_r(\Gamma_{\mathcal{Q}}, K_n) = \sum_{\mathcal{P} \preceq \mathcal{Q}} \omega_r(\mathcal{P}, n). \tag{2.1}$$

Recall that a collection of vertices W in a graph G is *independent* if there are no edges in G among the vertices in W. The following lemma provides a converse to Proposition 1.4 when computing weights.

Lemma 2.10. *Let η be a multihomomorphism of* $\mathrm{OHOM}(G_e, K_n)$*, and let $\mathcal{P}$ be the nonnesting partition whose arc diagram generates $U(\eta)$. Suppose $I = [a, c] \subseteq [m]$ is independent in $\Gamma_{\mathcal{P}}$. Then*

(1) $\eta(a) \cap \eta(c) \neq \emptyset$,
(2) $|\eta(a) \cap \eta(c)| = 1$, *and*
(3) *if $\eta(a) \cap \eta(c) = \{x_i\}$, then $\eta(b) = \{x_i\}$ for all $a < b < c$.*

Proof. To prove (1), suppose by way of contradiction that $\eta(a) \cap \eta(c) = \emptyset$. Consider the arc diagram $\Gamma_{\mathcal{Q}}$ obtained from $\Gamma_{\mathcal{P}}$ by adding the arc (a, c). Since I is independent in $\Gamma_{\mathcal{P}}$, the graph $\Gamma_{\mathcal{Q}}$ is the arc diagram of a nonnesting partition $\mathcal{Q}$.

First, we observe that $\mathcal{Q} \prec \mathcal{P}$ since $\Gamma_{\mathcal{P}}$ is a subgraph of $\Gamma_{\mathcal{Q}}$, and hence every arc of $\Gamma_{\mathcal{P}}$ lies above an arc of $\Gamma_{\mathcal{Q}}$. Next, we claim that $\eta \in \mathrm{OHOM}(\Gamma_{\mathcal{Q}}, K_n)$. Since (a, c) is the only edge in $E(\Gamma_{\mathcal{Q}}) \setminus E(\Gamma_{\mathcal{P}})$, we only need to check that (x, y) is an edge of K_n for any choice of $x \in \eta(a)$ and $y \in \eta(c)$. This follows immediately from our assumption that $\eta(a) \cap \eta(c) = \emptyset$.

Thus $\eta \in \mathrm{OHOM}(\Gamma_{\mathcal{Q}}, K_n)$ and $\mathcal{Q} \prec \mathcal{P}$, contradicting our assumption that the nonnesting partition $\mathcal{P}$ generates $U(\eta)$. This proves that $\eta(a) \cap \eta(c) \neq \emptyset$. Parts (2) and (3) follow immediately from the requirement that $\eta(a) \leq \eta(b) \leq \eta(c)$ for all $a < b < c$, together with the fact that $\eta(a) \cap \eta(c) \neq \emptyset$. □

Lemma 2.11. [2, Theorem 5.11] *If $\Gamma_{\mathcal{P}}$ contains an arc (a, c) where $c - a > 2$, then $\omega_r(\mathcal{P}, n) = 0$.*

Proof. Suppose to the contrary that $\Gamma_{\mathcal{P}}$ contains such an arc and that $\omega_r(\mathcal{P}, n) \neq 0$. Let η be an r-dimensional multihomomorphism of $\mathrm{OHOM}(G_e, K_n)$ such that $\Gamma_{\mathcal{P}}$ generates $U(\eta)$.

Consider the intervals $I = [a, c-1]$ and $I' = [a+1, c]$. Since $\mathcal{P}$ is nonnesting, I and I' are independent in $\Gamma_{\mathcal{P}}$. By Lemma 2.10, there is an element $x_i \in \eta(a) \cap \eta(c-1)$ and moreover,

$\eta(b) = \{x_i\}$ for all $a < b < c-1$. In particular, $\eta(a+1) = \{x_i\}$ since $a+1 < c-1$. By applying Lemma 2.10 to the interval I', we see that $\eta(a+1) \cap \eta(c) \neq \emptyset$ and hence $x_i \in \eta(c)$. Thus $x_i \in \eta(a) \cap \eta(c)$, which contradicts Proposition 1.4. $\square$

Following [2], we call an arc diagram $\Gamma_{\mathcal{P}}$ containing no arcs of the form (i,j) with $j-i > 2$ a *small arc diagram*, and we say that the corresponding nonnesting partition $\mathcal{P}$ is a *small nonnesting partition*. In light of Lemma 2.11, we need only compute the weights $\omega_r(\mathcal{P}, K_n)$ for which $\Gamma_{\mathcal{P}}$ is a small arc diagram. The following two results are interesting enumerative results in their own right.

Proposition 2.12. [5] *The number of nonnesting arc diagrams on* $[m]$ *is enumerated by the* m*-th Catalan number* $C_m = \frac{1}{m+1}\binom{2m}{m}$.

Proposition 2.13. [2, Theorem 5.12] *Let* F_m *denote the* m*th Fibonacci number with* $F_0 = F_1 = 1$. *The number of small arc diagrams on* $[m]$ *is* F_{2m-2}.

2.3. **An example.** As a more complicated example, we exhibit the weights and corresponding Betti numbers for all nonnesting partitions of $\{1,2,3\}$. We recall the arc diagrams $\Gamma_1, \ldots, \Gamma_5$ used in Figure 4.

Proposition 2.14. *For all* $r, n \geq 0$,

$$\omega_r(\Gamma_1, K_n) = \binom{n}{r+1}(r+1).$$

Proof. Let $\eta \in \mathrm{OHOM}(\Gamma_1, K_n)$ be a multihomomorphism whose upper order ideal $U(\eta)$ is generated by Γ_1. By Lemma 2.10, there is a single element $x_i \in \eta(1) \cap \eta(3)$ and $\eta(2) = \{x_i\}$. In order to compute $\omega_r(\Gamma_1, K_n)$, we first determine that there are $r+1$ distinct elements in $\eta(1) \cup \eta(2) \cup \eta(3)$. Indeed, by the inclusion-exclusion principle,

$$\begin{aligned}
|\eta(1) \cup \eta(2) \cup \eta(3)| &= |\eta(1)| + |\eta(2)| + |\eta(3)| \\
&\quad -|\eta(1) \cap \eta(2)| - |\eta(1) \cap \eta(3)| - |\eta(2) \cap \eta(3)| \\
&\quad +|\eta(1) \cap \eta(2) \cap \eta(3) \\
&= (r+3) - 3 + 1 \\
&= r+1.
\end{aligned}$$

In order to describe any such multihomomorphism η, we must choose a subset $X \subseteq \{x_1, \ldots, x_n\}$ of the $r+1$ distinct elements in $\eta(1) \cup \eta(2) \cup \eta(3)$, together with the single element $x_i \in X$ that is common to all three sets. Certainly there are $\binom{n}{r+1}(r+1)$ ways to make these choices. Having chosen X and $x_i \in X$, we take

$$\begin{aligned}
\eta(1) &= \{x \in X : x \leq x_i\}, \\
\eta(2) &= \{x_i\}, \text{ and} \\
\eta(3) &= \{x \in X : x \geq x_i\}.
\end{aligned}$$

$\square$

Proposition 2.15. *For all $r, n \geq 0$,*

$$\omega_r(\Gamma_2, K_n) = \binom{n}{r+1}\binom{r+1}{2}.$$

Proof. Let $\eta \in \mathrm{OHOM}(\Gamma_2, K_n)$ be an r-dimensional multihomomorphism whose upper order ideal $U(\eta)$ is generated by Γ_2. By Lemma 2.10, there is an element $x_i \in \eta(1) \cap \eta(2)$ and another element $x_j \in \eta(2) \cap \eta(3)$. Moreover, by Proposition 1.4, $\eta(1) \cap \eta(3) = \emptyset$ and hence $x_i \neq x_j$. Thus by the inclusion-exclusion principle, there are $r + 1$ distinct elements in $\eta(1) \cup \eta(2) \cup \eta(3)$.

In order to describe any such multihomomorphism η, we must first choose a subset $X \subseteq \{x_1, \ldots, x_n\}$ of the $r+1$ elements in $\eta(1) \cup \eta(2) \cup \eta(3)$, together with the elements $x_i \in \eta(1) \cap \eta(2)$ and $x_j \in \eta(2) \cap \eta(3)$. Certainly there are $\binom{n}{r+1}\binom{r+1}{2}$ ways to make these choices. Given the set X and distinguished elements x_i and x_j, we take

$$\begin{aligned}
\eta(1) &= \{x \in X : x \leq x_i\}, \\
\eta(2) &= \{x \in X : x_i \leq x \leq x_j\}, \text{ and} \\
\eta(3) &= \{x \in X : x \geq x_j\}.
\end{aligned}$$

□

Proposition 2.16. *For all $r, n \geq 0$,*

$$\omega_r(\Gamma_3, K_n) = \binom{n}{r+2}\binom{r+2}{2}.$$

Proof. Let $\eta \in \mathrm{OHOM}(\Gamma_3, K_n)$ be an r-dimensional multihomomorphism whose upper order ideal $U(\eta)$ is generated by Γ_3. By Lemma 2.10, there is an element $x_j \in \eta(2) \cap \eta(3)$, and by Proposition 1.4, $\eta(1) \cap \eta(2) = \emptyset$. By the inclusion-exclusion principle, there are $r + 2$ distinct elements in $\eta(1) \cup \eta(2) \cup \eta(3)$.

In order to describe any such multihomomorphism η, we must first choose a subset $X \subseteq \{x_1, \ldots, x_n\}$ of the $r + 2$ distinct elements in $\eta(1) \cup \eta(2) \cup \eta(3)$, together with the element $x_j \in \eta(2) \cap \eta(3)$ and the largest element x_i in $\eta(1)$. Certainly there are $\binom{n}{r+2}\binom{r+2}{2}$ ways to make these choices. As before, having chosen X, x_i and x_j, we take

$$\begin{aligned}
\eta(1) &= \{x \in X : x \leq x_i\}, \\
\eta(2) &= \{x \in X : x_i < x \leq x_j\}, \text{ and} \\
\eta(3) &= \{x \in X : x \geq x_j\}.
\end{aligned}$$

□

Proposition 2.17. *For all $r, n \geq 0$,*

$$\omega_r(\Gamma_4, K_n) = \binom{n}{r+2}\binom{r+2}{2}.$$

Proof. The proof of this proposition follows by an argument that is symmetric to the one given to compute the weights $\omega_r(\Gamma_3, K_n)$. □

Proposition 2.18. *For all $r, n \geq 0$,*

$$\omega_r(\Gamma_5, K_n) = \binom{n}{r+3}\binom{r+2}{2}.$$

Proof. Let $\eta \in \mathrm{OHOM}(\Gamma_5, K_n)$ be an r-dimensional multihomomorphism whose upper order ideal $U(\eta)$ is generated by Γ_5. By Proposition 1.4, $\eta(1) \cap \eta(2)$, $\eta(2) \cap \eta(3)$, and $\eta(1) \cap \eta(3)$ are empty. Thus by the inclusion-exclusion principle, $|\eta(1) \cup \eta(2) \cup \eta(3)| = r + 3$.

In order to describe such a multihomomorphism η, we must choose a subset $X \subseteq \{x_1, \ldots, x_n\}$ of the $r + 3$ distinct elements of $\eta(1) \cup \eta(2) \cup \eta(3)$ together with the maximal elements x_i and x_j of $\eta(1)$ and $\eta(2)$ respectively. Having made these choices, we take

$$\begin{aligned} \eta(1) &= \{x \in X : x \leq x_i\}, \\ \eta(2) &= \{x \in X : x_i < x \leq x_j\}, \text{ and} \\ \eta(3) &= \{x \in X : x > x_j\}. \end{aligned}$$

Since $\eta(3)$ must be nonempty, we cannot choose x_j to be the maximal element of X. The number of ways to choose X, x_i, and x_j is $\binom{n}{r+3}\binom{r+2}{2}$, which completes the proof. $\square$

3. Enumerative results

Our goal for this section is to prove the promised formula computing the weights $\omega_r(\mathcal{P}, n)$ for any small nonnesting partition $\mathcal{P}$. Before stating the main theorem, we establish notation that will be used for the remainder of the paper.

Proposition 3.1. *For any small nonnesting partition $\mathcal{P}$ of $[m]$, there is a unique constant $k = k(\mathcal{P})$ and a unique decomposition of $[m]$ into intervals $I_1, \ldots, I_k$ satisfying the following conditions.*

- (P1): $I_1 \cup \cdots \cup I_k = [m]$,
- (P2): $I_1 \leq I_2 \leq \cdots \leq I_k$,
- (P3): $|I_j| \geq 2$ *for all j, and*
- (P4): *each interval I_j satisfies exactly one of the following conditions:*
    - (i). *I_j is a maximal interval (under inclusion) that is independent in $\Gamma_{\mathcal{P}}$, or*
    - (ii). *$I_j = \{i_j, i_{j+1}\}$ and (i_j, i_{j+1}) is an edge of $\Gamma_{\mathcal{P}}$.*

Proof. We induct on m. The result is clear when $m = 2$. When $m \geq 3$, we examine two cases.

If $(1, 2)$ is an arc in $\Gamma_{\mathcal{P}}$, let $I_1 = \{1, 2\}$. Inductively, we may decompose the restriction of $\mathcal{P}$ to $[2, m]$ into intervals $I_2, \ldots, I_k$ satisfing conditions (P1)-(P4).

On the other hand, if $(1, 2)$ is not an arc in $\Gamma_{\mathcal{P}}$, let t be the largest element of $[m]$ such that $[1, t]$ is independent in $\Gamma_{\mathcal{P}}$. Let $I_1 = [1, t]$; if $t = m$, we have found the desired decomposition. Otherwise, if $t < m$, the restriction of $\Gamma_{\mathcal{P}}$ to $[t, m]$ is a small arc diagram, and we may inductively decompose the restriction of $\Gamma_{\mathcal{P}}$ to $[t, m]$ into intervals $I_2, \ldots, I_k$ satisfying conditions (P1)-(P4).

In either of the above cases, we must check that the resulting interval decomposition $[m] = I_1 \cup \cdots \cup I_k$ satisfies conditions (P1)-(P4). Conditions (P1)-(P3) are satisfied by the inductive hypothesis. We must check, however, that if I_1 and I_2 are both edgefree as in condition (P4.i), then both are maximal under inclusion. By our construction, $I_1 = [1, t]$ is maximal. Since $t+1 \notin I_1$

and $\mathcal{P}$ is small, either $(t, t+1)$ or $(t-1, t+1)$ is an edge in $\Gamma_{\mathcal{P}}$. If $(t, t+1)$ is an edge in $\Gamma_{\mathcal{P}}$, then $I_2 = \{t, t+1\}$ satisfies condition (P4.ii). If $(t-1, t+1)$ is an edge in $\Gamma_{\mathcal{P}}$, then I_2 satisfies condition (P4.i), and $t-1$ cannot be added to I_2 without violating the independence condition. Thus I_2 is maximal under inclusion, which completes the proof. $\square$

Example 3.2. Consider the small arc diagram $\Gamma_{\mathcal{P}}$ shown in Figure 5.

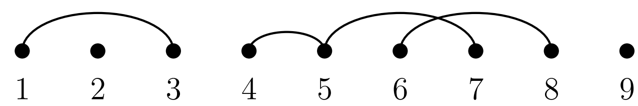

FIGURE 5. The arc diagram for $\mathcal{P} = \{\{1,3\},\{4,5,7\},\{6,8\},\{9\}\}$.

The interval decomposition of $\Gamma_{\mathcal{P}}$ is

$$\begin{aligned} I_1 &= \{1,2\} \\ I_2 &= \{2,3,4\} \\ I_3 &= \{4,5\} \\ I_4 &= \{5,6\} \\ I_5 &= \{6,7\} \\ I_6 &= \{7,8,9\}. \end{aligned}$$

Theorem 3.3. *Let $\mathcal{P}$ be a small nonnesting partition of $[m]$ with interval decomposition $I_1, \ldots, I_k$ as described by Proposition 3.1. For any $r, n \geq 0$,*

$$\omega_r(\mathcal{P}, n) = \begin{cases} \binom{n}{\ell}\binom{\ell-1}{k} & \text{if } (1,2),(m-1,m) \in E(\Gamma_{\mathcal{P}}); \\ \binom{n}{\ell}\binom{\ell}{k} & \text{otherwise,} \end{cases} \tag{3.1}$$

where $\ell := r + m - \sum_{j \in J}(|I_j| - 1)$ and $J \subseteq [k]$ indexes those intervals described by condition (P4.i).

Proof. Fix a small nonnesting partition $\mathcal{P}$ of $[m]$. For each $1 \leq j \leq k$, let $I_j = [a_j, c_j]$. For any r-dimensional multihomomorphism $\eta \in \mathrm{OHOM}(\Gamma_e, K_n)$, we observe that $\sum_{i=1}^{m} |\eta(i)| = r + m$. If the arc diagram for $\Gamma_{\mathcal{P}}$ generates $U(\eta)$, then Lemma 2.10 prescribes the combinatorial structure of the intersections of the sets $\eta(i)$ within each interval $I_1, \cdots, I_k$.

As a consequence of these Lemmas, we claim that as a *set*,

$$\ell := |\eta(1) \cup \cdots \cup \eta(m)| = r + m - \sum_{j \in J}(|I_j| - 1),$$

where $J \subseteq [k]$ indexes those intervals described by condition (P4.i). To see this, we simply observe that for each interval I_j with $j \in J$, there is a single element x_j common to the sets among $\{\eta(p) : p \in I_j\}$. When computing $|\eta(1) \cup \cdots \cup \eta(m)|$, each of these elements x_j is overcounted $|I_j| - 1$ times.

Thus in order to describe such a multihomomorphism η, we must first choose a subset $X \subseteq \{x_1, \ldots, x_n\}$ of the ℓ distinct elements of $\eta(1) \cup \cdots \cup \eta(m)$. This can be accomplished in $\binom{n}{\ell}$ ways.

Now suppose that $(1, 2)$ is not an arc of $\Gamma_{\mathcal{P}}$. The binomial coefficient $\binom{\ell}{k}$ counts the number of ways in which we may decompose the set X into pairwise disjoint intervals $A_0 < A_1 < \cdots < A_k$ so that the sets $A_1, \ldots, A_k$ are nonempty. This follows from a standard stars-and-bars argument [5, Section 1.2] by arranging the elements of X linearly as

$$x_{i_1} \quad x_{i_2} \quad \cdots \quad x_{i_{\ell-1}} \quad x_{i_\ell},$$

with $i_1 < \cdots < i_\ell$ and choosing k of the spaces between consecutive elements of X to partition the set. This includes the possibility of choosing the space to the left of x_{i_1}, which corresponds to the case that A_0 is empty.

We now exhibit a bijection between the family of stars-and-bars partitions of X described in the previous paragraph and the collection of multihomomorphisms $\eta \in \mathrm{OHOM}(G_e, K_n)$ such that $\eta(1) \cup \cdots \cup \eta(m) = X$ and $\mathcal{P}$ generates $U(\eta)$.

Given pairwise disjoint intervals $A_0 < A_1 < \cdots < A_k$ that partition X with $A_1, \ldots, A_k$ nonempty, let m_i denote the smallest element of A_i for $1 \leq i \leq k$. We determine the sets $\eta(i)$ by declaring that

- $A_0 \subseteq \eta(1)$,
- $A_j \subseteq \eta(c_j)$ for all $1 \leq j \leq k$, and
- $m_j \in \eta(b)$ for all $b \in [a_j, c_j]$ and all $j \in J$.

Lemma 2.10 and Proposition 1.4 show that this is a bijective correspondence. By symmetry, the same argument applies to the situation that $(m-1, m) \notin \Gamma_{\mathcal{P}}$.

In the case that both $(1, 2)$ and $(m-1, m)$ are edges in $\Gamma_{\mathcal{P}}$, an analogous bijection holds, with the exception that $\binom{\ell-1}{k}$ counts the number of partitions of X into nonempty, pairwise disjoint intervals $B_0 < \cdots < B_k$. Here we must require that B_0 and B_k are nonempty as they describe the elements of $\eta(1)$ and $\eta(m)$ respectively. $\square$

We illustrate the proof of Theorem 3.3 in the following example.

Example 3.4.

Let $\mathcal{P}$ be the small partition from Example 3.2. Suppose $\ell = 11$ and (for simplicity) that $\eta(1) \cup \cdots \cup \eta(9) = \{x_1, \ldots, x_{11}\}$. The stars-and-bars decomposition

$$x_1 \quad x_2 \quad | \quad x_3 \quad | \quad x_4 \quad x_5 \quad x_6 \quad | \quad x_7 \quad | \quad x_8 \quad | \quad x_9 \quad | \quad x_{10} \quad x_{11}$$

$\eta(1)$	$\eta(2)$	$\eta(3)$	$\eta(4)$	$\eta(5)$	$\eta(6)$	$\eta(7)$	$\eta(8)$	$\eta(9)$
x_1								
x_2								
x_3	x_3							
	x_4	x_4	x_4					
			x_5					
			x_6					
				x_7				
				x_8	x_8			
					x_9	x_9		
						x_{10}	x_{10}	x_{10}
								x_{11}

TABLE 3. The multihomomorphism η described by Example 3.4

gives

$$\begin{aligned}
A_0 &= \{x_1, x_2\},\\
A_1 &= \{x_3\},\\
A_2 &= \{x_4, x_5, x_6\},\\
A_3 &= \{x_7\},\\
A_4 &= \{x_8\},\\
A_5 &= \{x_9\},\\
A_6 &= \{x_{10}, x_{11}\}.
\end{aligned}$$

This, in turn corresponds to the multihomomorphism η shown in Table 3. We have shaded the blocks $A_j \subseteq \eta(c_j)$ for all $1 \leq j \leq 6$, where the intervals $I_1, \ldots, I_6$ are those given in Example 3.2 and we write $I_j = [a_j, c_j]$ as in the proof of Theorem 3.3.

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