Bertrand's postulate over the Gaussian integers

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1 Introduction

Bertrand's postulate states that for each positive integer, n, there exists a prime p such that $n \leq p \leq 2n$. This statement was first proposed by Bertrand in 1845, and he verified this conjecture for all $n < 3 \times 10^6$. The first proof of Bertrand's postulate was given by Chebyshev in 1852 using analytic methods. In 1919, Ramanujan [7] gave a simpler proof, and finally Erdős [1, 4] gave a completely elementary proof in 1932.

In this paper, we propose an extension of Bertrand's postulate to the Gaussian integers. We begin by reviewing standard notation and definitions. The *Gaussian integers*, denoted $\mathbb{Z}[\mathbf{i}]$, are the set of all integers of the form $a + b\mathbf{i}$ with $a, b \in \mathbb{Z}$ and $\mathbf{i}^2 = -1$. The *units* are the Gaussian integers ± 1, $\pm\mathbf{i}$, and the *associates* of a Gaussian integer are its unit multiples. The *norm* of a Gaussian integer is $N(a + b\mathbf{i}) := a^2 + b^2$. A Gaussian integer ρ is *prime* if, whenever $\rho \mid \alpha \cdot \beta$, either $\rho \mid \alpha$ or $\rho \mid \beta$. Equivalently, since $\mathbb{Z}[\mathbf{i}]$ is a unique factorization domain, ρ is prime if its only divisors are ± 1, $\pm\mathbf{i}$, $\pm\rho$, and $\pm\mathbf{i}\rho$. The prime Gaussian integers are classified by the following result, which is standard (see [8]):

Theorem 1.1. *A Gaussian integer $\rho = a + b\mathbf{i}$ is prime if and only if either*

1. *$N(\rho)$ is prime (i.e., $\rho = \pm(1 \pm \mathbf{i})$ or $N(\rho) \equiv 1 \bmod 4$ is prime), or*
2. *$b = 0$ and $|a| \equiv 3 \bmod 4$ is prime or $a = 0$ and $|b| \equiv 3 \bmod 4$ is prime.*

Colloquially, Bertrand's postulate is often explained by saying "there is always a prime between n and $2n$." In order to extend Bertrand's postulate to the Gaussian integers, we must answer two additional questions. First, what does it mean to say that one Gaussian integer lies between two others? Second, what does it mean to double a Gaussian integer?

Typically, the norm is used as a rough measure of whether one Gaussian integer is bigger or smaller than another, and hence it seems reasonable to say that a Gaussian integer γ lies between Gaussian integers α and β if $N(\alpha) < N(\gamma) < N(\beta)$. However, the region bounded by $N(\alpha)$ and $N(\beta)$ is an annulus, and it makes little intuitive sense to say that γ is between α and β when it may lie in an entirely different region of the plane from α and β. So, we propose an additional condition to account for the relative positions of α and β.

Definition 1.2. Let $\alpha = a + b\mathbf{i}$ and $\beta = c + d\mathbf{i}$ be Gaussian integers with $N(\alpha) < N(\beta)$. We define the *set of Gaussian integers between* α *and* β, denoted $\mathcal{B}(\alpha, \beta)$, to be the set of all Gaussian integers $x + y\mathbf{i}$ such that

1. $\min(a, c) \leq x \leq \max(a, c)$,
2. $\min(b, d) \leq y \leq \max(b, d)$, and
3. $N(\alpha) \leq x^2 + y^2 \leq N(\beta)$.

Geometrically, $\mathcal{B}(\alpha, \beta)$ can be viewed as the intersection of the rectangle whose sides are determined by the real and imaginary parts of α and β with the annulus whose inner radius is $\sqrt{N(\alpha)}$ and whose outer radius is $\sqrt{N(\beta)}$. This is illustrated in Figure 1.

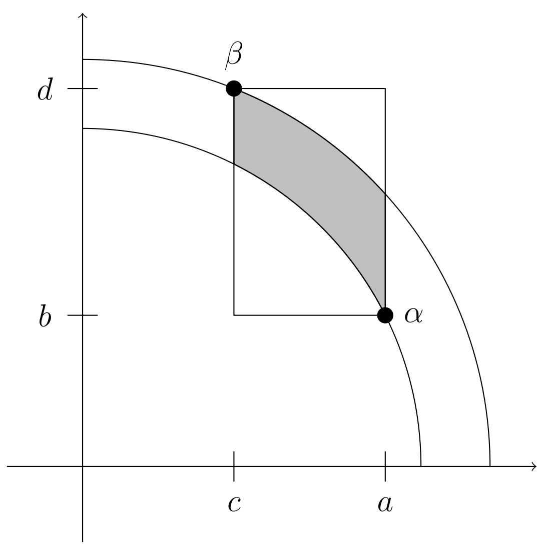

Figure 1: The region $\mathcal{B}(\alpha, \beta)$.

Second, we must answer the question of how to double a Gaussian integer. The obvious answer to this question is to say that 2α is the double of α. However, it is also reasonable to say that $(1 + \mathbf{i})\alpha$ is the double of α since multiplication by $1 + \mathbf{i}$ doubles the norm. We explore both of these doubling methods in this note.

Therefore, we propose the following extensions of Bertrand's postulate to the Gaussian integers. Just as the classical version of Bertrand's postulate only considers positive integers, we only consider those Gaussian integers that lie in the first quadrant, which is the set of Gaussian integers whose real part is strictly positive and whose imaginary part is nonnegative.

Conjecture 1.3. *For all $\alpha = a + b\mathbf{i} \in \mathbb{Z}[\mathbf{i}]$ with $a > 0$, $b \geq 0$, and $\alpha \neq 1$, there exists a Gaussian prime ρ such that $\rho \in \mathcal{B}(\alpha, 2\alpha)$.*

Conjecture 1.4. *For all $\alpha = a + b\mathbf{i} \in \mathbb{Z}[\mathbf{i}]$ with $a > 0$ and $b \geq 0$, there exists a Gaussian prime ρ such that $\rho \in \mathcal{B}(\alpha, (1 + \mathbf{i})\alpha)$.*

Note that if $\alpha = a + b\mathbf{i}$ with $a, b \geq 1$ and $\rho = x + y\mathbf{i}$ is a Gaussian prime such that $a \leq x \leq 2a$ and $b \leq y \leq 2b$, then $\rho \in \mathcal{B}(\alpha, 2\alpha)$ since the condition that $N(\alpha) \leq x^2 + y^2 \leq N(2\alpha)$ is automatically satisfied. Geometrically, the rectangle determined by α and 2α is always contained within the annulus bounded by the norms of α and 2α, so the norm condition is redundant.

In the remainder of the paper, we explore these two notions of doubling, either as multiplication by 2 (in Section 2) or as as multiplication by $1 + \mathbf{i}$ (in Section 3). In both cases, we provide experimental results showing Conjectures 1.3 and 1.4 hold for small Gaussian integers—in particular, we verify Conjecture 1.3 for all Gaussian integers $a + b\mathbf{i}$ with $1 \leq a, b \leq 10^{100}$. We include our code and relevant data in Appendix A.

Furthermore, we give asymptotic results showing Conjectures 1.3 and 1.4 hold for sufficiently large Gaussian integers. In both cases, the asymptotic results require that we understand the number of Gaussian primes in a circular sector. This analysis only applies to the Gaussian integers whose real and imaginary parts are both strictly positive. We settle Conjecture 1.3 for Gaussian integers on the real axis; however, Conjecture 1.4 seems profoundly difficult for the Gaussian integers on the real axis. Instead, we verify Conjecture 1.4 for all Gaussian integers $a + 0\mathbf{i}$ with $a \leq 10^8$ experimentally.

2 Doubling as multiplication by 2

We begin by exploring Bertrand's postulate when interpreting doubling as multiplication by 2.

2.1 Analysis for small Gaussian primes

Our first goal is to verify Conjecture 1.3 computationally for small Gaussian integers. We are able to do this for all Gaussian integers $a+b\mathbf{i}$ with $1 \leq a, b \leq 10^{100}$. It is computationally infeasible to check each of these Gaussian integers. Instead we make use of the fact that a single Gaussian prime lies between α and 2α for a large number of Gaussian integers α.

Lemma 2.1. *Let $\rho = x + y\mathbf{i}$ be a Gaussian prime with $x, y \geq 1$. Then Conjecture 1.3 holds for all Gaussian integers $\alpha = a + b\mathbf{i}$ with $\lceil \frac{x}{2} \rceil \leq a \leq x$ and $\lceil \frac{y}{2} \rceil \leq b \leq y$.*

Proof: Notice that $\rho \in \mathcal{B}(\alpha, 2\alpha)$ for all such α since $a \leq x \leq 2a$ and $b \leq y \leq 2b$. The norm condition is clearly satisfied. $\square$

Lemma 2.2. *If Conjecture 1.3 holds for $\alpha = a + b\mathbf{i}$, then it also holds for $\alpha' = b + a\mathbf{i}$.*

Proof: There exists a Gaussian prime $\rho = x + y\mathbf{i} \in \mathcal{B}(\alpha, 2\alpha)$, and hence $\rho' = y + x\mathbf{i}$ is a Gaussian prime that belongs to $\mathcal{B}(\alpha', 2\alpha')$. $\square$

In particular, Lemma 2.2 allows us to reduce Conjecture 1.3 to those Gaussian integers $a + b\mathbf{i}$ such that $0 < b \leq a$.

Lemma 2.3. *Let y and N be positive integers. Assume that there exists a sequence of Gaussian primes $x_1 + y\mathbf{i},\ x_2 + y\mathbf{i},\ \ldots,\ x_t + y\mathbf{i}$ such that*

(1). $x_1 < x_2 < \cdots < x_{t-1} < N \leq x_t$, and

(2). $x_{j+1} \leq 2x_j$ for all $1 \leq j < t$.

Then Conjecture 1.3 holds for each Gaussian integer $a + b\mathbf{i}$ such that $\frac{x_1}{2} \leq a \leq N$ and $\frac{y}{2} \leq b \leq y$.

Proof: Let $\alpha = a + b\mathbf{i}$ be a Gaussian integer satisfying $\frac{x_1}{2} \leq a \leq N$ and $\frac{y}{2} \leq b \leq y$.

If $\frac{x_1}{2} \leq a \leq x_1$, then $x_1 + y\mathbf{i} \in \mathcal{B}(\alpha, 2\alpha)$ by Lemma 2.1.

Otherwise, $a > x_1$ and there exists an index $1 \leq j < t$ such that $x_j < a \leq x_{j+1}$. But this implies that $\frac{x_{j+1}}{2} < a \leq x_{j+1}$ because $x_j \geq \frac{x_{j+1}}{2}$ by condition (2) above. Therefore, $x_{j+1} + y\mathbf{i} \in \mathcal{B}(\alpha, 2\alpha)$ by Lemma 2.1. $\square$

In particular, if $x_1 \leq y \leq x_2$ in Lemma 2.3, then Conjecture 1.3 holds for each Gaussian integer $a + b\mathbf{i}$ such that $\frac{y}{2} \leq a \leq N$ and $\frac{y}{2} \leq b \leq y$.

Proposition 2.4. *Let $\alpha = a + b\mathbf{i} \in \mathbb{Z}[\mathbf{i}]$ with $1 \leq b \leq 10^{100}$ and $b \leq a \leq 10^{100}$. Then $\mathcal{B}(\alpha, 2\alpha)$ contains a Gaussian prime.*

Proof: Rather than blindly verify Conjecture 1.3 in each of these approximately $\frac{10^{200}}{2}$ cases, we use Lemma 2.1 to find a relatively small set of primes with the property that one of these primes lies in $\mathcal{B}(\alpha, 2\alpha)$ for each α under consideration.

We present the following greedy algorithm for finding such a set of primes.

Step 1: Set $x = 10^{100}$ and $y = 10^{100}$.

Step 2:
- (a) Initialize an empty list, P, of found primes.
- (b) Find the smallest integer $k \geq 0$ such that $(x + k) + y\mathbf{i}$ is prime.
- (c) Append $(x + k) + y\mathbf{i}$ to P.
- (d) Set $x \to \lceil \frac{x+k}{2} \rceil$.
- (e) Repeat until $x = \lceil \frac{x+k}{2} \rceil$ or $x < \frac{y}{2}$.

(f) Save the list P.

Step 3: Set $y \to \lceil \frac{y}{2} \rceil$ and $x \to 10^{100}$. Repeat Step 2 until $y = 1$.

Several comments are in order.

First of all, it is unknown whether Step 2(b) will terminate for arbitrary x and y. In fact, if Step 2(b) were to terminate for arbitrary $x + y\mathbf{i}$, then Landau's fourth problem would be solved. However, for these small values of x and y, we were always successful in finding such a prime such that k was relatively small.

In Step 2(f), the set of primes $P = \{x_1 + y\mathbf{i}, x_2 + y\mathbf{i}, \ldots, x_t + y\mathbf{i}\}$ is sorted according to their real parts so that $x_1 < x_2 < \cdots < x_t$. These integers satisfy the conditions of Lemma 2.3 since $x_j \geq \lceil \frac{x_{j+1}}{2} \rceil$. In Step 2(e), we impose the halting condition when $x < \frac{y}{2}$ to cut down on computational time. This will guarantee that $x_1 \leq y \leq x_2$. Otherwise, the condition that $x = \lceil \frac{x+k}{2} \rceil$ states that the algorithm cannot proceed beyond the given x-value because $x + y\mathbf{i}$ is the first Gaussian prime encountered to the right of $\lceil \frac{x}{2} \rceil + y\mathbf{i}$.

We implemented this algorithm in Sage [9] with $N = 10^{100}$. For each y, the corresponding value of x_1 is listed online [5]. The data for small values of y is available in Table A.2. In particular, we have $x_1 \leq y$ for all $5 \leq y \leq 10^{100}$, and hence Conjecture 1.3 holds for all Gaussian integers $a + b\mathbf{i}$ with $3 = \left\lceil \frac{5}{2} \right\rceil \leq b \leq a \leq 10^{100}$.

When $y = 3$, the smallest Gaussian prime in P is $8 + 3\mathbf{i}$. Therefore, Lemma 2.3 implies that Conjecture 1.3 holds for all Gaussian integers $a + b\mathbf{i}$ with $4 \leq a \leq 10^{100}$ and $2 \leq b \leq 3$. The only Gaussian integers $a + b\mathbf{i}$ with $2 \leq b \leq a \leq 10^{100}$ not counted here are $2 + 2\mathbf{i}$ and $3 + 2\mathbf{i}$.

When $y = 2$, the smallest Gaussian prime in P is $3 + 2\mathbf{i}$. Therefore, Lemma 2.3 implies that Conjecture 1.3 holds for all Gaussian integers $a + b\mathbf{i}$ with $2 \leq a \leq 10^{100}$ and $1 \leq b \leq 2$. In particular, the exceptional integers $2 + 2\mathbf{i}$ and $3 + 2\mathbf{i}$ from the previous case are covered here.

Therefore, the only Gaussian integer $a + b\mathbf{i}$ with $1 \leq b \leq a \leq 10^{100}$ that is not included in the above data is $1 + \mathbf{i}$. It is clear that Conjecture 1.3 holds for $1 + \mathbf{i}$ as $1 + \mathbf{i}$ is prime. This completes the proof.

□

The following result follows immediately from Lemma 2.2

Corollary 2.5. *Let $\alpha = a + b\mathbf{i}$ with $1 \leq a, b \leq 10^{100}$. Then $\mathcal{B}(\alpha, 2\alpha)$ contains a Gaussian prime.*

2.2 Analysis when $\mathrm{Im}(\alpha) = 0$

In this section, we consider Gaussian integers α with $\mathrm{Im}(\alpha) = 0$. If $\alpha = a + 0\mathbf{i}$, $\mathcal{B}(\alpha, 2\alpha) = \{c + 0\mathbf{i} \ : \ a \leq c \leq 2a\}$.

Cullinan and Farshid [3] showed that for all $x \geq 7$, the interval $(x, 2x]$ contains a prime congruent to 1 mod 4 and a prime congruent to 3 mod 4. In fact, one can verify by hand that for all $x \geq 2$, the interval $[x, 2x]$ contains a prime congruent to 3 mod 4. The following result is immediate.

Proposition 2.6. *For all $a \geq 2$, there exists a Gaussian prime in $\mathcal{B}(a+0\mathbf{i}, 2a+0\mathbf{i})$.*

2.3 Asymptotic results

Since we have already established Conjecture 1.3 for Gaussian integers on the real axis (and hence also for their associates on the imaginary axis), we now turn our attention to Conjecture 1.3 for Gaussian integers $a+b\mathbf{i}$ with $a, b \geq 1$.

We make use of the following result of Kubilyus [6], which approximates the number of Gaussian primes that lie in a given sector of a circle.

Theorem 2.7. [6], [2, Theorem 2.1]
Let D be an angular sector of the circle $x^2+y^2 \leq R^2$ with angle θ ($0 < \theta < \frac{\pi}{2}$). Then the number of Gaussian primes in D, denoted $\Pi_\theta(R)$, is given by

$$\Pi_\theta(R) = \frac{\theta}{\pi/2} \cdot \frac{R^2}{\log(R^2)} + O\left(\frac{R^2}{\log^2(R)}\right).$$

Moreover, constant in the error term does not depend on θ.

We aim to establish Conjecture 1.3 for sufficiently large Gaussian integers. Our approach is first to find a polar rectangle contained in $\mathcal{B}(\alpha, 2\alpha)$ for a Gaussian integer α in the first octant, and second to use Theorem 2.7 to show that this polar rectangle contains a Gaussian prime once $N(\alpha)$ is sufficiently large. In light of this, we write $\Pi_\theta(R_1, R_2; \eta)$ to denote the number of Gaussian primes, ρ, such that $R_1^2 < N(\rho) \leq R_2^2$ and $\eta \leq \arg(\rho) \leq \eta+\theta$.

Theorem 2.8. *Conjecture 1.3 holds for all Gaussian integers α for which $N(\alpha)$ is sufficiently large and* $\arg(\alpha)$ *is bounded away from all of 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π.*

Proof: Without loss of generality, we may limit our attention to the first octant of the complex plane. Accordingly, let $\alpha = a+b\mathbf{i}$ be a Gaussian integer with $\tan(\arg(\alpha)) = \frac{b}{a} = m < 1$. In this way, we may view α as a point in the Euclidean plane lying on the line $y = mx$. Next, define $\theta = \arctan(m) - \arctan\left(\frac{3}{4}m\right)$, $\eta = \arctan\left(\frac{3}{4}m\right)$, $R_1^2 = \frac{16}{9}a^2 + b^2$, and $R_2^2 = 4a^2 + \frac{9}{4}b^2$.

The angle θ is bounded between the lines through the origin of slope $\frac{3}{4}m$ and m. The former line passes through the points $\frac{4}{3}a + b\mathbf{i}$ and $2a + \frac{3}{2}b\mathbf{i}$, whose norms are R_1^2 and R_2^2, respectively. The latter line passes through the points $a+b\mathbf{i}$ and $2(a+b\mathbf{i})$. Since $a^2+b^2 \leq R_1^2 \leq R_2^2 \leq (2a)^2 + (2b)^2$, the polar rectangle with radius bounded between R_1 and R_2 and with argument bounded between η and $\eta+\theta$ is contained within $\mathcal{B}(\alpha, 2\alpha)$. This is illustrated in Figure 2 below.

We claim that $\Pi_\theta(R_1, R_2; \eta) > 0$ whenever $N(\alpha)$ and $\arg(\alpha)$ are sufficiently large. By Theorem 2.7, there exists a constant c such that

$$\frac{\theta}{\pi/2} \cdot \frac{R_1^2}{\log(R_1^2)} - c \cdot \frac{R_1^2}{\log^2(R_1^2)} \leq \Pi_\theta(R_1) \leq \frac{\theta}{\pi/2} \cdot \frac{R_1^2}{\log(R_1^2)} + c \cdot \frac{R_1^2}{\log^2(R_1^2)}$$

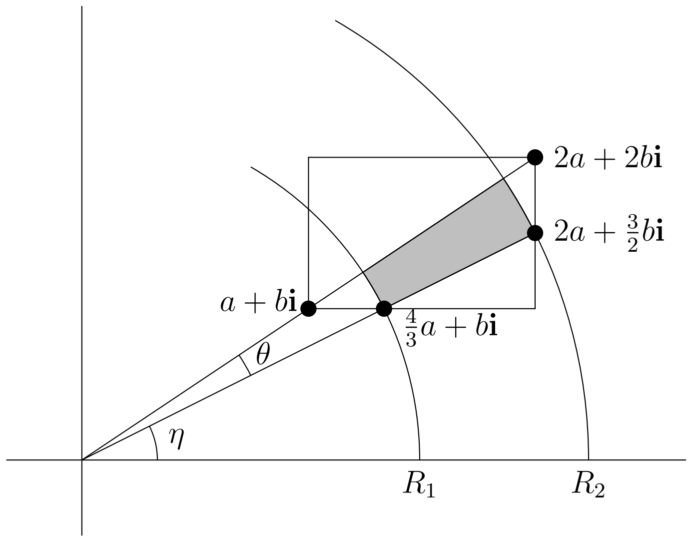

Figure 2: A polar rectangle contained in $\mathcal{B}(\alpha, 2\alpha)$.

and

$$\frac{\theta}{\pi/2} \cdot \frac{R_2^2}{\log(R_2^2)} - c \cdot \frac{R_2^2}{\log^2(R_2^2)} \leq \Pi_\theta(R_2) \leq \frac{\theta}{\pi/2} \cdot \frac{R_2^2}{\log(R_2^2)} + c \cdot \frac{R_2^2}{\log^2(R_2^2)}.$$

Therefore,

$$\begin{aligned}
\Pi_\theta(R_1, R_2; \eta) &= \Pi_\theta(R_2) - \Pi_\theta(R_1) \\
&\geq \frac{\theta}{\pi/2} \cdot \frac{R_2^2}{\log(R_2^2)} - c \cdot \frac{R_2^2}{\log^2(R_2^2)} - \left(\frac{\theta}{\pi/2} \cdot \frac{R_1^2}{\log(R_1^2)} + c \cdot \frac{R_1^2}{\log^2(R_1^2)} \right) \\
&= \frac{\theta}{\pi/2} \left(\frac{R_2^2}{\log(R_2^2)} - \frac{R_1^2}{\log(R_1^2)} \right) - c \left(\frac{R_2^2}{\log^2(R_2^2)} + \frac{R_1^2}{\log^2(R_1^2)} \right).
\end{aligned}$$

Using the fact that $R_2^2 = \frac{9}{4} R_1^2$ and applying logarithm rules, the final expression above is equal to

$$\frac{\theta}{\pi/2} \left(\frac{\frac{9}{4} R_1^2}{2 \log R_2} - \frac{R_1^2}{2 \log R_1} \right) - c \left(\frac{\frac{9}{4} R_1^2}{4 \log^2 R_2} + \frac{R_1^2}{4 \log^2 R_1} \right).$$

Since we may safely assume that $R_1 \geq \left(\frac{3}{2}\right)^{29/7}$ (approximately 5.36), it follows that $R_2 = \frac{3}{2} R_1 \leq R_1^{36/29}$, so that the above expression is greater than or equal to

$$\frac{\theta}{\pi/2} \left(\frac{\frac{9}{4} R_1^2}{2 \cdot \frac{36}{29} \log R_1} - \frac{R_1^2}{2 \log R_1} \right) - c \left(\frac{\frac{9}{4} R_1^2}{4 \log^2 R_1} + \frac{R_1^2}{4 \log^2 R_1} \right).$$

Simplification yields the following:

$$\begin{aligned}\Pi_\theta(R_1, R_2; \eta) &\geq \left[\frac{2\theta}{\pi}\left(\frac{29}{32} - \frac{1}{2}\right) - c\left(\frac{9}{16\log R_1} + \frac{1}{4\log R_1}\right)\right]\frac{R_1^2}{\log R_1} \\ &\geq \left[\frac{13\theta}{16\pi} - \frac{13c}{16}\cdot\frac{1}{\log R_1}\right]\frac{R_1^2}{\log R_1}.\end{aligned}$$

To complete the proof, we need only show that the expression in square brackets is eventually positive. Note that this occurs exactly when $\frac{\theta}{\pi} > \frac{c}{\log R_1}$, or equivalently, $R_1^\theta > \exp(\pi c)$. This inequality defines a wedge-shaped region in the first quadrant, which can be seen in Figure 3 below.

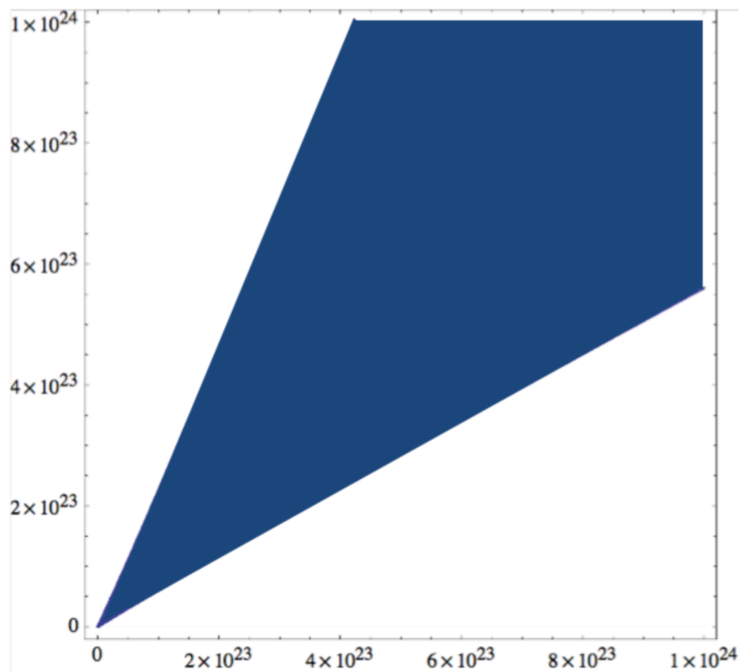

Figure 3: Theorem 2.8's region of validity for $c = 2$.

Observing that $R_1 > \sqrt{N(\alpha)}$ and $\theta > \frac{9}{50}\arg(\alpha)$ for α in the first octant, it follows that Conjecture 1.3 holds for the less strict but more elegant condition $N(\alpha)^{\arg(\alpha)} > \exp\left(\frac{100\pi c}{9}\right)$. The eightfold symmetry of the Gaussian primes gives the result for all octants. $\square$

3 Doubling as multiplication by $1 + \mathbf{i}$.

3.1 Analysis for small Gaussian primes

It is not possible to match the computational efficiency outlined in the proof of Proposition 2.4 when doubling as multiplication by $1 + \mathbf{i}$. Instead, we were forced to verify Conjecture

1.4 individually for each Gaussian integer within a given range. The code for verifying the following result can be found in Appendix A and is available for download online [5].

Proposition 3.1. *Let* $\alpha = a + b\mathbf{i}$ *with* $1 \le a, b \le 10^5$. *Then* $\mathcal{B}(\alpha, (1+\mathbf{i})\alpha)$ *contains a Gaussian prime.*

3.2 Analysis when $\mathrm{Im}(\alpha) = 0$

In this section, we consider Gaussian integers α with $\mathrm{Im}(\alpha) = 0$. If $\alpha = a + 0\mathbf{i}$, then $\mathcal{B}(\alpha, (1+i)\alpha) = \{a + b\mathbf{i} \ : \ 0 \le b \le a\}$. In general, the problem of determining whether or not there exists a Gaussian prime on every vertical line in the complex plane (without any condition on the size of its imaginary part relative to its real part) seems incredibly difficult. Instead, we provide computational evidence supporting Conjecture 1.4.

Proposition 3.2. *For all* $1 \le a \le 10^8$, *the region* $\mathcal{B}(a + 0\mathbf{i}, (1+\mathbf{i})(a+0\mathbf{i}))$ *contains a Gaussian prime.*

This result is perhaps made more interesting by the plots in Figures 5 and 6 in Appendix A. For each a, the plot shows the first Gaussian prime $a + b\mathbf{i}$ with real part a (i.e., the Gaussian prime $a + b\mathbf{i}$ with $b \ge 1$ minimal). We note that, except for the Gaussian primes $1+\mathbf{i}$ and $17449677+598\mathbf{i}$, the first Gaussian prime $a+b\mathbf{i}$ with real part a satisfies $b < \log_2(a)^2$ for all $a \le 10^8$, which is considerably stronger than the bound predicted by Conjecture 1.4. For comparison, $\log_2(17449677)^2 \approx 578.72$. Figures 5 and 6 in Appendix A show the first Gaussian prime with real part a for $a \le 10^6$ and $a \le 10^8$, along with the curve $f(x) = \log_2(x)^2$.

3.3 Asymptotic results

Once again, we will use Theorem 2.7, this time to establish Conjecture 1.4 asymptotically.

Theorem 3.3. *Conjecture 1.4 holds for all Gaussian integers* α *for which* $N(\alpha)$ *is sufficiently large and* $\arg(\alpha)$ *is bounded away from all of* 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, *and* 2π.

Proof: Without loss of generality, we may limit our attention to the first quadrant of the complex plane.[1] Let $\alpha = a + b\mathbf{i}$ be a Gaussian integer in the first quadrant with $\tan(\arg(\alpha)) = \frac{b}{a} = m \le 4$. Next, define $\theta = \arctan(\frac{3}{4}+m) - \arctan(\frac{1}{2}+m)$, $\eta = \arctan(\frac{1}{2}+m)$, $R_1^2 = a^2 + b^2$, and $R_2^2 = \frac{5}{4}R_1^2$. Again, we consider the polar rectangle with radius bounded between R_1 and R_2 and with argument bounded between η and $\eta + \theta$, as shown in Figure 4.

We claim that the given polar rectangle is contained within the region $\mathcal{B}(\alpha, (1+\mathbf{i})\alpha)$. First note that the angle θ is constructed so that it is bounded between the lines through the origin of slope $\frac{1}{2} + m$ and $\frac{3}{4} + m$, which pass through the line $\mathrm{Re}(z) = a$ at the points

[1] Unlike Theorem 2.8, we may not restrict to the first octant, since $\mathcal{B}(a+b\mathbf{i}, (1+i)(a+b\mathbf{i}))$ is not symmetric to $\mathcal{B}(b + a\mathbf{i}, (1+i)(b+a\mathbf{i}))$ about the line $y = x$.

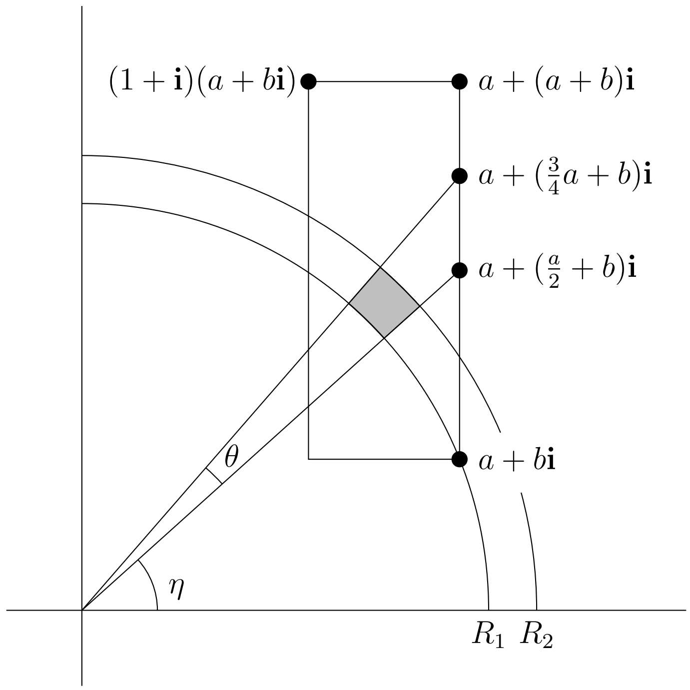

Figure 4: A polar rectangle contained in $\mathcal{B}(\alpha,(1+\mathbf{i})\alpha)$.

$a+(\frac{a}{2}+b)\mathbf{i}$ and $a+(\frac{3}{4}a+b)\mathbf{i}$ respectively. Since $m \leq 4$, it follows that $b \leq 4a$. Thus the norm of $a+(\frac{a}{2}+b)\mathbf{i}$ is bounded below by R_2^2:

$$N\left(a+\left(\frac{a}{2}+b\right)\mathbf{i}\right)=\frac{5}{4}a^2+ab+b^2 \geq \frac{5}{4}a^2+\frac{1}{4}b^2+b^2=\frac{5}{4}\left(a^2+b^2\right)=R_2^2.$$

It follows that the given polar rectangle is contained within the region between α and $(1+\mathbf{i})\alpha$.

As before, Theorem 2.7 ensures that

$$\Pi_\theta(R_1,R_2;\eta) \quad \geq \quad \frac{\theta}{\pi/2}\left(\frac{R_2^2}{\log(R_2^2)}-\frac{R_1^2}{\log(R_1^2)}\right)-c\left(\frac{R_2^2}{\log^2(R_2^2)}+\frac{R_1^2}{\log^2(R_1^2)}\right).$$

This time, we use the fact that $R_2^2=\frac{5}{4}R_1^2$ to get

$$\Pi_\theta(R_1,R_2;\eta) \quad \geq \quad \frac{\theta}{\pi/2}\left(\frac{\frac{5}{4}R_1^2}{2\log R_2}-\frac{R_1^2}{2\log R_1}\right)-c\left(\frac{\frac{5}{4}R_1^2}{4\log^2 R_2}+\frac{R_1^2}{4\log^2 R_1}\right).$$

Since we may safely assume that $R_1 \geq \left(\frac{\sqrt{5}}{2}\right)^{73/7}$ (approximately 3.20), it follows that $R_2 = \frac{\sqrt{5}}{2}R_1 \leq R_1^{80/73}$, so that the above expression is greater than or equal to

$$\frac{\theta}{\pi/2}\left(\frac{\frac{5}{4}R_1^2}{2\cdot\frac{80}{73}\log R_1}-\frac{R_1^2}{2\log R_1}\right)-c\left(\frac{\frac{5}{4}R_1^2}{4\log^2 R_1}+\frac{R_1^2}{4\log^2 R_1}\right).$$

Simplification yields the following:

$$\begin{aligned}\Pi_\theta(R_1, R_2; \eta) &\geq \left[\frac{2\theta}{\pi}\left(\frac{73}{128} - \frac{1}{2}\right) - c\left(\frac{5}{16\log R_1} + \frac{1}{4\log R_1}\right)\right]\frac{R_1^2}{\log R_1} \\ &\geq \left[\frac{\theta}{\pi}\cdot\frac{9}{64} - \frac{9c}{16}\cdot\frac{1}{\log R_1}\right]\frac{R_1^2}{\log R_1}.\end{aligned}$$

The expression in square brackets is positive exactly when $R_1^\theta \geq \exp(4\pi c)$. So, the result holds for α in the first quadrant, subject to the conditions that R_1 is sufficiently large, θ is bounded away from zero, and $\arg(\alpha) = \arctan(m) \leq \arctan(4) \approx 1.3258$. By symmetry, this may be extended to all four quadrants. As before, the conditions may be recast in terms of $N(\alpha)$ and $\arg(\alpha)$ only: specifically, $N(\alpha)^{\arg(\alpha)} > \exp(100\pi c)$ and $\arg(\alpha) \leq \arctan(4)$. □

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A Supplemental Code

The following code was used to verify Conjectures 1.3 and 1.4 for small values of α. The source files are also available online [5].

The first piece of code checks whether a given Gaussian integer is prime. We enter a Gaussian integer $\alpha = a + b\mathbf{i}$ as an ordered pair (a, b) for simplicity.

```
def PrimeCheck(a,b):
    if not(a==0 or b==0):
        return is_prime(a^2+b^2)
    if a==0 and abs(b)%4==3:
        return is_prime(b)
    if b==0 and abs(a)%4==3:
        return is_prime(a)
```

A.1 Doubling as multiplication by $1 + \mathbf{i}$

The first piece of code checks for a prime between a given $\alpha = a + b\mathbf{i}$ and $(1 + \mathbf{i})\alpha = (a - b) + (a + b)\mathbf{i}$. The code assumes that $a, b \geq 0$ so that $a - b \leq a$ and $a + b \geq b$.

```
def PrimeBox(a,b):
    for x in [a-b..a]:
        for y in [b..a+b]:
```

```
            if a^2+b^2 <= x^2+y^2 <= 2*(a^2+b^2):
                if PrimeCheck(x,y): return (x,y)
    return -1
```

This code simply tests each Gaussian integer $x+y\mathbf{i}$ with $a-b \leq x \leq a$ and $b \leq y \leq a+b$ (i.e., those Gaussian integers within the rectangular box determined by α and $(1+\mathbf{i})\alpha$) until it finds a Gaussian prime. To cut down on computational time, we first test that the norm condition $N(\alpha) \leq x^2+y^2 \leq N((1+\mathbf{i})\alpha)$ is satisfied, then we check to see if $x+y\mathbf{i}$ is a Gaussian prime. If no Gaussian prime is found, the code returns -1.

The next piece of code inputs a value of N and verifies Conjecture 1.4 for all Gaussian integers $a+b\mathbf{i}$ with $1 \leq a, b \leq N$.

```
def ConjVerify1(N):
    for a in [1..N]:
        for b in [1..N]:
            if PrimeBox(a,b) == -1:
                print (a,b)
                return false
    return true
```

We implemented this code with $N = 10^5$. The output verified Conjecture 1.4 for these values.

Finally, we wrote code to verify Conjecture 1.4 for all Gaussian integers $a+0\mathbf{i}$ with $1 \leq a \leq N$.

```
def PrimeLine(N):
    FirstPrimeList = []
    for a in [1..N]:
        b = 1
        primeFound = false
        while not(primeFound) and b<=a:
            if is_prime(a^2+b^2):
                primeFound = true
                FirstPrimeList.append((a,b))
            else:
                b+=1
        if not(primeFound):
            print a
    return FirstPrimeList
```

For each value $1 \leq a \leq N$, the list FirstPrimeList stores the coordinates of the Gaussian prime $a+b\mathbf{i}$ with minimal b. If such a Gaussian prime with $b \leq a$ is not found, the value of a is printed. For $N = 10^8$, no such counterexamples were found. The data from FirstPrimeList for $N = 10^6$ and $N = 10^8$ are shown in Figures 5 and 6. In Figure 6, we see the exceptional prime at $17449677 + 598\mathbf{i}$.

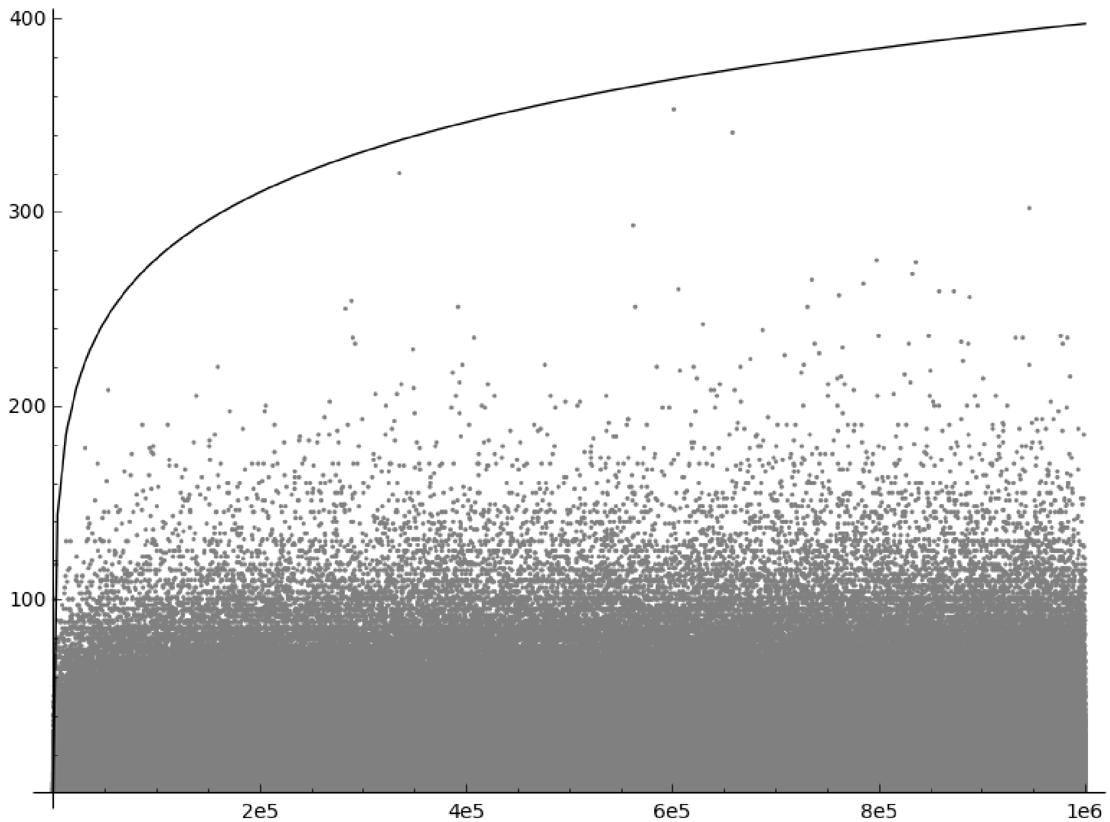

Figure 5: The first Gaussian prime with real part a for $a \leq 10^6$.

A.2 Doubling as multiplication by 2

Here is the code that implements the algorithm used in Proposition 2.4.

```
def PrimeBox(N):
    y = N
    while y>1:
        x = N
        while x >= y/2:
            k=0
            primeFound=false
            while not(primeFound):
                if is_prime((x+k)^2+y^2):
                    primeFound=true
                k+=1
            if x == ceil((x+k)/2):
                x=0
            else:
```

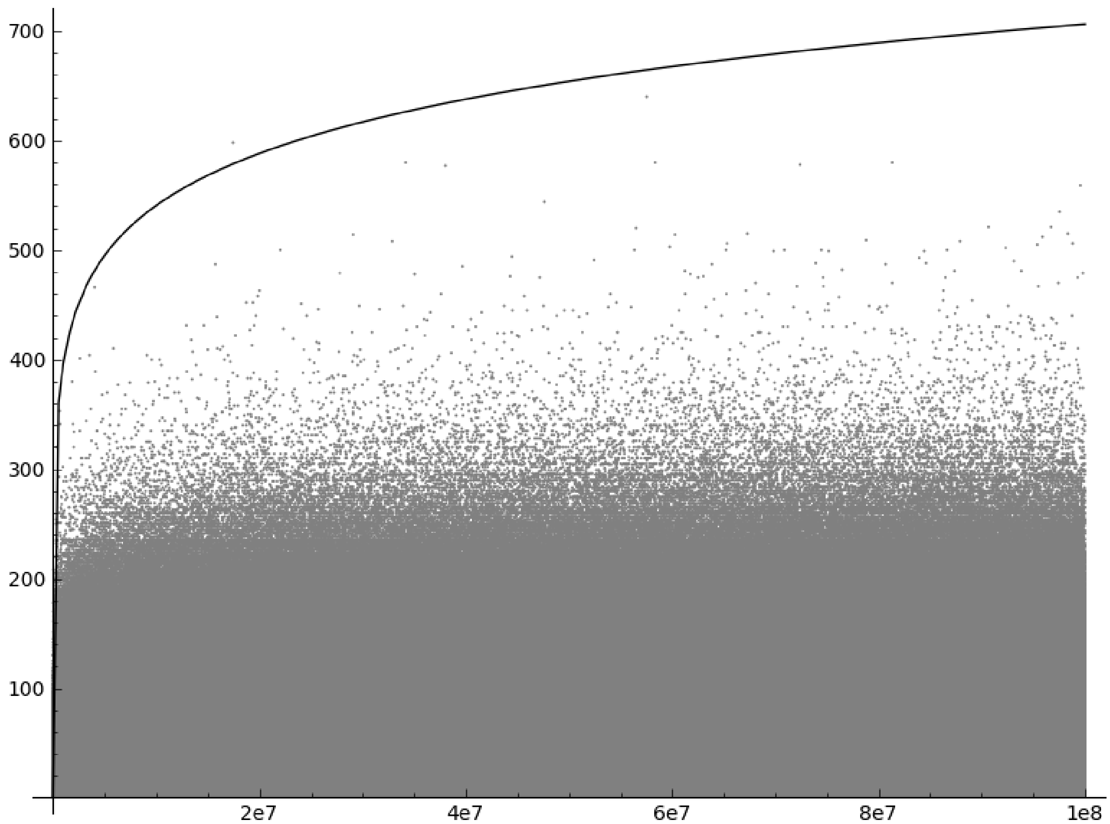

Figure 6: The first Gaussian prime with real part a for $a \leq 10^8$.

```
        x = ceil((x+k)/2)
    print (x,y)
    y = ceil(y/2)
```

For each value of x and y, the code begins by finding the first k such that $(x+k)+y\mathbf{i}$ is prime. Since we avoid the coordinate axes, we need only check that the norm is prime. If $x = \lceil \frac{x+k}{2} \rceil$, the algorithm halts as $k = x$. We set $x = 0$ here because it will guarantee that $x < \frac{y}{2}$ at the next pass through the inner while loop. In the notation of Proposition 2.4, the command to print (x, y) shows the value of the prime $x_1 + y\mathbf{i}$ for each given y value. It is implicit from the code that the primes $x_1 + y\mathbf{i}, x_2 + y\mathbf{i}, \ldots, x_t + y\mathbf{i}$ satisfying the conditions of Lemma 2.3 exist. The code is initialized at $y = N$ and proceeds as long as $y > 1$.

The output of this code for all values of y with $N = 10^{100}$ are available online at [5]. The values for $N = 10^{100}$ and $y \leq 10^{10}$ are shown in Table A.2.

y	$x_1 + y\mathbf{i}$	$\frac{y}{2} < x_1 < y$?
613636684	$306818419 + 613636684\mathbf{i}$	yes
306818342	$153409213 + 306818342\mathbf{i}$	yes
153409171	$76704660 + 153409171\mathbf{i}$	yes
76704586	$38352319 + 76704586\mathbf{i}$	yes
38352293	$19176180 + 38352293\mathbf{i}$	yes
19176147	$9588140 + 19176147\mathbf{i}$	yes
9588074	$4794121 + 9588074\mathbf{i}$	yes
4794037	$2397050 + 4794037\mathbf{i}$	yes
2397019	$1198534 + 2397019\mathbf{i}$	yes
1198510	$599283 + 1198510\mathbf{i}$	yes
599255	$299664 + 599255\mathbf{i}$	yes
299628	$149873 + 299628\mathbf{i}$	yes
149814	$74969 + 149814\mathbf{i}$	yes
74907	$37540 + 74907\mathbf{i}$	yes
37454	$18759 + 37454\mathbf{i}$	yes
18727	$9380 + 18727\mathbf{i}$	yes
9364	$4709 + 9364\mathbf{i}$	yes
4682	$2365 + 4682\mathbf{i}$	yes
2341	$1196 + 2341\mathbf{i}$	yes
1171	$610 + 1171\mathbf{i}$	yes
586	$309 + 586\mathbf{i}$	yes
293	$168 + 293\mathbf{i}$	yes
147	$118 + 147\mathbf{i}$	yes
74	$49 + 74\mathbf{i}$	yes
37	$28 + 37\mathbf{i}$	yes
19	$14 + 19\mathbf{i}$	yes
10	$7 + 10\mathbf{i}$	yes
5	$4 + 5\mathbf{i}$	yes
3	$8 + 3\mathbf{i}$	no
2	$3 + 2\mathbf{i}$	yes

Table 1: Primes $x_1 + y\mathbf{i}$ for $N = 10^{100}$ and $y \leq 10^{10}$ output by the code PrimeBox

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