PRIME LABELING OF FAMILIES OF TREES WITH GAUSSIAN INTEGERS

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Abstract. A graph on n vertices is said to admit a prime labeling if we can label its vertices with the first n natural numbers such that any two adjacent vertices have relatively prime labels. Here we extend the idea of prime labeling to the Gaussian integers, which are the complex numbers whose real and imaginary parts are both integers. We begin by defining an order on the Gaussian integers that lie in the first quadrant. Using this ordering, we show that several families of trees admit a prime labeling with the Gaussian integers.

1. Introduction

A graph on n vertices admits a **prime labeling** if its vertices can be labeled with the first n natural numbers in such a way that any two adjacent vertices have relatively prime labels. Many families of graphs are known to admit prime labelings – such as paths, stars, caterpillars, complete binary trees, spiders, palm trees, fans, flowers, and many more [1], [2]. Entringer conjectured that any tree admits a prime labeling, however this conjecture has not been proven for all trees. In this paper we extend the study of prime labelings to the Gaussian integers.

In order to extend the notion of a prime labeling to Gaussian integers, we must first define what we mean by "the first n Gaussian integers." In Section 2, we define a spiral ordering on the Gaussian integers that allows us to linearly order the Gaussian integers. This spiral ordering preserves many familiar properties of the natural ordering on $\mathbb{N}$. For example, the spiral ordering alternates parity, and consecutive odd integers in the spiral ordering are relatively prime. We discuss further properties of the spiral ordering in Section 2. In Section 3, we apply the properties of the spiral ordering to prove that several families of trees admit a prime labeling with the Gaussian integers under the spiral ordering.

2. Background and Definitions

2.1. Background on Gaussian Integers.

We begin with some relevant background on Gaussian integers to provide a foundation for our work.

The **Gaussian integers**, denoted $\mathbb{Z}[\mathbf{i}]$, are the complex numbers of the form $a+b\mathbf{i}$, where $a, b \in \mathbb{Z}$ and $\mathbf{i}^2 = -1$. A **unit** in the Gaussian integers is one of $\pm 1, \pm \mathbf{i}$. An **associate** of a Gaussian integer α is $u \cdot \alpha$ where u is a Gaussian unit. The **norm** of a Gaussian integer $a+b\mathbf{i}$, denoted by $N(a+b\mathbf{i})$,

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is given by $a^2 + b^2$. A Gaussian integer is **even** if it is divisible by $1 + \mathbf{i}$ and **odd** otherwise. This is because Gaussian integers with even norms are divisible by $1 + \mathbf{i}$.

Definition 2.1. A Gaussian integer, π, is **prime** if its only divisors are ± 1, $\pm\mathbf{i}$, $\pm\pi$, or $\pm\pi\mathbf{i}$.

Besides this definition of Gaussian primes, we have the following characterization theorem for Gaussian primes. Further information on the Gaussian integers can be found in Rosen's *Elementary Number Theory* [3].

Theorem 2.2. *A Gaussian integer $\alpha \in \mathbb{Z}[\mathbf{i}]$ is prime if and only if either*

- $\alpha = \pm 1 \pm \mathbf{i}$,
- $N(\alpha)$ *is a prime integer congruent to* $1 \bmod 4$, *or*
- $\alpha = p + 0\mathbf{i}$ *or* $\alpha = 0 + p\mathbf{i}$ *where* p *is a prime in* $\mathbb{Z}$ *and* $|p| \equiv 3 \bmod 4$.

Definition 2.3. Let α be a Gaussian integer and let β be a Gaussian integer. We say α and β are **relatively prime** or **coprime** if their only common divisors are the units in $\mathbb{Z}[\mathbf{i}]$.

2.2. **Background on Graphs.** We also need some definitions relating to graphs before we can dive into Gaussian prime labeling.

Definition 2.4. A **graph** $G = (V, E)$ consists of a finite, nonempty set V of **vertices** and a set E of unordered pairs of distinct vertices called **edges**. If $\{u, v\} \in E$, we say u and v are connected by an edge and write $uv \in E$ for brevity.

A graph is commonly represented as a diagram representing a collection of dots (vertices) connected by line segments (edges).

Definition 2.5. The **degree** of a vertex is the number of edges incident to that vertex.

Definition 2.6. A **tree** is a connected graph that contains no cycles.

Definition 2.7. An **internal node** of a tree is any vertex of degree greater than 1. A **leaf** or **endvertex** of a tree is a vertex of degree 1.

For more information on graph theory, we refer to Trudeau's *Introduction to Graph Theory* [4].

2.3. **Prime Labeling with Gaussian Integers.** Our goal is to extend the study of the prime labeling of trees to the Gaussian integers. Because the Gaussian integers are not totally ordered, we must first give an appropriate definition of "the first n Gaussian integers." We propose the following ordering:

Definition 2.8. The **spiral ordering of the Gaussian integers** is a recursively defined ordering of the Gaussian integers. We denote the n^{th} Gaussian integer in the spiral ordering by γ_n. The ordering is defined beginning with $\gamma_1 = 1$ and continuing as:

$$\gamma_{n+1} = \begin{cases} \gamma_n + \mathbf{i}, & \text{if } \operatorname{Re}(\gamma_n) \equiv 1 \bmod 2,\ \operatorname{Re}(\gamma_n) > \operatorname{Im}(\gamma_n) + 1 \\ \gamma_n - 1, & \text{if } \operatorname{Im}(\gamma_n) \equiv 0 \bmod 2,\ \operatorname{Re}(\gamma_n) \leq \operatorname{Im}(\gamma_n) + 1,\ \operatorname{Re}(\gamma_n) > 1 \\ \gamma_n + 1, & \text{if } \operatorname{Im}(\gamma_n) \equiv 1 \bmod 2,\ \operatorname{Re}(\gamma_n) < \operatorname{Im}(\gamma_n) + 1 \\ \gamma_n + \mathbf{i}, & \text{if } \operatorname{Im}(\gamma_n) \equiv 0 \bmod 2,\ \operatorname{Re}(\gamma_n) = 1 \\ \gamma_n - \mathbf{i}, & \text{if } \operatorname{Re}(\gamma_n) \equiv 0 \bmod 2,\ \operatorname{Re}(\gamma_n) \geq \operatorname{Im}(\gamma_n) + 1,\ \operatorname{Im}(\gamma_n) > 0 \\ \gamma_n + 1, & \text{if } \operatorname{Re}(\gamma_n) \equiv 0 \bmod 2,\ \operatorname{Im}(\gamma_n) = 0 \end{cases}$$

This is illustrated below.

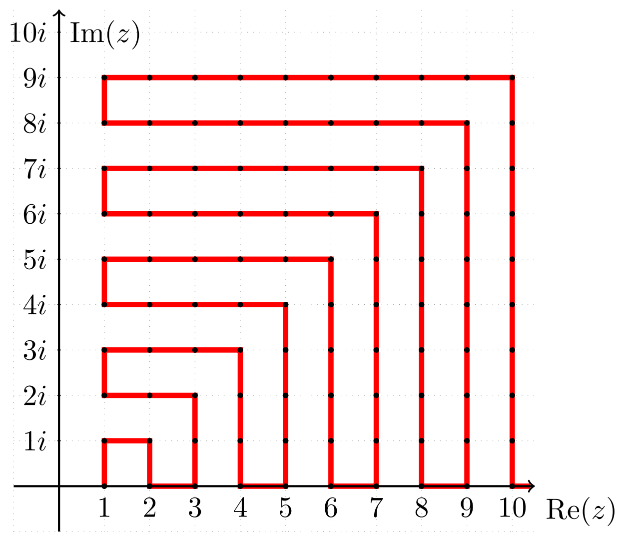

Figure 1. Spiral Ordering of Gaussian Integers

Under this ordering, the first 10 Gaussian integers are

$$1,\ 1+\mathbf{i},\ 2+\mathbf{i},\ 2,\ 3,\ 3+\mathbf{i},\ 3+2\mathbf{i},\ 2+2\mathbf{i},\ 1+2\mathbf{i},\ 1+3\mathbf{i},\ \ldots,$$

and we write $[\gamma_n]$ to denote the set of the first n Gaussian integers in the spiral ordering.

We exclude the imaginary axis to ensure that the spiral ordering excludes associates. Consecutive Gaussian integers in this ordering are separated by a unit and therefore alternate parity, as in the usual ordering of $\mathbb{N}$. However, several properties of the ordinary integers do not hold. In the set of the first $N(1+2\mathbf{i}) \cdot k$ numbers, $(k \in \mathbb{N})$, it is not guaranteed that there are exactly k multiples of $1+2\mathbf{i}$ (or any other residue class mod $1+2\mathbf{i}$). Furthermore, odd integers with indices separated by a power of two are not guaranteed to be relatively prime to each other.

This definition of the spiral ordering for the Gaussian integers leads to the following definition of prime labeling of trees with Gaussian integers.

Definition 2.9. A **Gaussian prime labeling** of a graph G on n vertices is a labeling of the vertices of G with the first n Gaussian integers in the spiral ordering such that if two vertices are adjacent, their labels are relatively prime. When it is necessary, we view the labeling as a bijection $\ell: V(G) \to [\gamma_n]$.

2.4. **Properties of the Spiral Ordering.** We define several pieces of the Gaussian spiral ordering. **Corners** of the spiral ordering occur when the spiral turns from north to east or east to north, from south to east or east to south, or from north to west or west to north. **Branches** of the spiral occur when the spiral travels along a straight path going north, south, east, or west. Steps along the real axis and the $\mathrm{Re}(z) = 1$ line are not counted as branches.

Our first goal is to determine the index of an arbitrary Gaussian integer, $a + b\mathbf{i}$, in the spiral order based on which type of branch or corner it lies on. We use $I(a + b\mathbf{i})$ to denote the index of $a + b\mathbf{i}$ in the spiral ordering. First note that there are three types of corners:

- *real corners* at Gaussian integers on the real axis,
- $\mathrm{Re}(z) = 1$ *corners* at Gaussian integers on the $\mathrm{Re}(z) = 1$ line, and
- *interior corners* at Gaussian integers on the line $\mathrm{Re}(z) - \mathrm{Im}(z) = 1$.

Gaussian integers at real corners are even when $\mathrm{Re}(z)$ is even and is odd otherwise. Gaussian integers at $\mathrm{Re}(z) = 1$ corners are even when $\mathrm{Im}(z)$ is odd and are even otherwise.

Similarly the branches come in four types:

- *up-oriented branches*, which contain Gaussian integers between odd corners on the real axis and interior corners,
- *down-oriented branches*, which contain Gaussian integers between interior corners and even corners on the real axis,
- *right-oriented branches*, which contain Gaussian integers between even corners on the $\mathrm{Re}(z) =$ 1 line and interior corners, and
- *left-oriented branches*, which contain Gaussian integers between interior corners and odd corners on the $\mathrm{Re}(z) = 1$ line.

Lemma 2.10. *Corners in the spiral ordering lie on either the real axis, the* $\mathrm{Re}(z) = 1$ *line, or the line* $\mathrm{Im}(z) = \mathrm{Re}(z) - 1$. *Their indices are found through the following equations:*

$$I(a+b\mathbf{i}) = \begin{cases} a^2, & \text{if } b = 0,\ a \equiv 0 \bmod 2 \text{ — Even corners on the real axis} \\ (a-1)^2 + 1, & \text{if } b = 0,\ a \equiv 1 \bmod 2 \text{ — Odd corners on the real axis} \\ (b+1)^2, & \text{if } a = 1,\ b \equiv 0 \bmod 2 \text{ — Odd corners on the } \mathrm{Re}(z) = 1 \text{ line} \\ b^2 + 1, & \text{if } a = 1,\ b \equiv 1 \bmod 2 \text{ — Even corners on the } \mathrm{Re}(z) = 1 \text{ line} \\ (a-1)^2 - a, & \text{if } b = a - 1 \text{ — Interior corners} \end{cases}$$

Proof. Observe that Gaussian integers of the form $a + 0\mathbf{i}$, with a even, or $1 + b\mathbf{i}$, with b even, are the corner nodes in squares composed of a^2 or $(b+1)^2$ nodes respectively. The spiral-ordering path will pass through each of these nodes once and will end on an even corner on the real axis if the number of nodes is even, and an odd corner on the $\mathrm{Re}(z) = 1$ line if the number of nodes is odd. Therefore, the index of an even corner on the real or an odd corner on the $\mathrm{Re}(z) = 1$ line will be the number of nodes in that square.

Odd corners on the real axis and even corners on the $\mathrm{Re}(z) = 1$ line will always have the index following the corresponding even corners of the real axis and odd corners on the $\mathrm{Re}(z) = 1$ line.

The interior corners are of the form $(b + 1) + b\mathbf{i}$. If b is even, then the corner is b nodes before an odd corner on the $\mathrm{Re}(z) = 1$ line. If b is odd, the corner is b nodes before an even corner on the $\mathrm{Re}(z) = 1$ line. In either case, since $a = b + 1$, the index of the corner will be $a^2 - b$. □

Theorem 2.11. *Let $a + b\mathbf{i}$ be a Gaussian integer with $a > 0$ and $b \geq 0$. Then its index in the spiral ordering, $I(a + b\mathbf{i})$, is given by the following formula:*

$$I(a+b\mathbf{i}) = \begin{cases} (a-1)^2 + 1 + b, & \text{if } a \equiv 1 \bmod 2,\ a \geq (b+1) \text{ — Up-oriented branches} \\ (b+1)^2 - a + 1, & \text{if } b \equiv 0 \bmod 2,\ a \leq (b+1) \text{ — Left-oriented branches} \\ b^2 + a, & \text{if } b \equiv 1 \bmod 2,\ a \leq (b+1) \text{ — Right-oriented branches} \\ a^2 - b, & \text{if } a \equiv 0 \bmod 2,\ a \geq (b+1) \text{ — Down-oriented branches} \end{cases}$$

Proof. Each branch references the corners of the spiral ordering. If $a + b\mathbf{i}$ lies on an up-oriented branch, then a is odd. Consider the odd corner on the real axis at $a + 0\mathbf{i}$. By Lemma 2.10, the index of this corner is $(a - 1)^2 + 1$. Therefore the index of $a + b\mathbf{i}$ is $(a - 1)^2 + 1 + b$.

If $a + b\mathbf{i}$ lies on a left-oriented branch, then b is even. Consider the odd corner on the $\mathrm{Re}(z) = 1$ line at $1 + b\mathbf{i}$. By Lemma 2.10, the index of this corner is $(b + 1)^2$. Therefore the index of $a + b\mathbf{i}$ is $(b + 1)^2 - (a - 1) = (b + 1)^2 - a + 1$.

If $a + b\mathbf{i}$ lies on a right-oriented branch, then b is odd. Consider the even corner on the $\mathrm{Re}(z) = 1$ line at $1 + b\mathbf{i}$. By Lemma 2.10, the index of this corner is $b^2 + 1$. Therefore the index of $a + b\mathbf{i}$ is $b^2 + 1 + (a - 1) = b^2 + a$.

If $a + b\mathbf{i}$ lies on a down-oriented branch, then a is even. Consider the even corner on the real axis at $a + 0\mathbf{i}$. By Lemma 2.10, the index of this corner is a^2. Therefore the index of $a + b\mathbf{i}$ is $a^2 - b$. □

Now that we have a formula for the index of a Gaussian integer in the spiral ordering, we prove several lemmas about Gaussian integers that will be useful in proving that various families of trees have Gaussian prime labelings.

Lemma 2.12. *Let α be a Gaussian integer and u be a unit. Then α and $\alpha + u$ are relatively prime.*

Proof. Suppose that there exists a Gaussian integer λ such that $\lambda | \alpha$ and $\lambda | (\alpha + u)$. This means that λ must also divide $u = (\alpha + u) - \alpha$. But the only Gaussian integers that divide u are the units, so λ must be a unit. Thus α and $\alpha + u$ are relatively prime. □

The following corollary is immediate because consecutive Gaussian integers in the spiral ordering have a difference of one unit.

Corollary 2.13. *Consecutive Gaussian integers in the spiral ordering are relatively prime.*

Lemma 2.14. *Let α be an odd Gaussian integer, let c be a positive integer, and let u be a unit. Then α and $\alpha + u \cdot (1+\mathbf{i})^c$ are relatively prime.*

Proof. Suppose that α and $\alpha + u \cdot (1+\mathbf{i})^c$ share a common divisor γ. It follows that γ divides $u \cdot (1+\mathbf{i})^c = (\alpha + u \cdot (1+\mathbf{i})^c) - \alpha$. However, the only divisors of $u \cdot (1+\mathbf{i})^c$ in $\mathbb{Z}[\mathbf{i}]$ are $(1+\mathbf{i})$ or its associates and the units in $\mathbb{Z}[\mathbf{i}]$. Since α is odd, it is not divisible $(1+\mathbf{i})$ or its associates because those numbers are all even. Therefore, γ must be a unit. Hence α and $\alpha + u \cdot (1+\mathbf{i})^c$ are relatively prime. □

Corollary 2.15. *Consecutive odd Gaussian integers in the spiral ordering are relatively prime.*

Proof. Consecutive odd Gaussian integers in the spiral ordering differ by two units. The only possible differences between them are therefore $1+\mathbf{i}$, 2, or one of their associates. Since $2 = -\mathbf{i}(1+\mathbf{i})^2$, all of these differences are of the form $u \cdot (1+\mathbf{i})^c$ so the result follows from Lemma 2.14. □

Lemma 2.16. *Let α be a Gaussian integer and let π be a prime Gaussian integer. Then α and $\alpha + \pi$ are relatively prime if and only if $\pi \nmid \alpha$.*

Proof. Assume that there exists a Gaussian integer γ such that $\gamma \mid \alpha$ and $\gamma \mid \alpha + \pi$. Then γ must also divide $(\alpha + \pi) - \alpha = \pi$. But π is prime, so either $\gamma = u$ or $\gamma = \pi \cdot u$ for some unit u. If $\gamma = \pi \cdot u$, then α and $\alpha + \pi$ have a common factor of π and are not relatively prime. If γ is a unit, then α and $\alpha + \pi$ have only a common factor of a unit and are relatively prime. Therefore α and $\alpha + \pi$ are relatively prime if and only if $\pi \nmid \alpha$. □

Lemma 2.17. *Let $p \in \mathbb{N}$ be a prime integer congruent to $3 \bmod 4$ and let $\alpha = a + b\mathbf{i}$ be a Gaussian integer such that $p \mid \alpha$. Then the index of α in the spiral ordering is congruent to 0 or $2 \bmod p$.*

Proof. By Theorem 2.2, p is prime in $\mathbb{Z}[\mathbf{i}]$, so if $p \mid a + b\mathbf{i}$, then $p \mid a$ and $p \mid b$. So $a \equiv b \equiv 0 \bmod p$. From Theorem 2.11, we can calculate the index $I(\alpha)$ of α and examine it modulo p. There are four cases to consider. In Case 1, $I(\alpha) = (a-1)^2 + 1 + b \equiv (-1)^2 + 1 + 0 \bmod p \equiv 2 \bmod p$. In Case 2, $I(\alpha) = (b+1)^2 - a + 1 \equiv 1^2 - 0 + 1 \bmod p \equiv 2 \bmod p$. In Case 3, $I(\alpha) = b^2 + a \equiv 0^2 + 0 \bmod p \equiv 0 \bmod p$. In Case 4, $I(\alpha) = a^2 - b \equiv 0^2 - 0 \bmod p \equiv 0 \bmod p$. Therefore $I(\alpha) \equiv 0 \bmod p$ or $I(\alpha) \equiv 2 \bmod p$ for any α such that $p \mid \alpha$. □

3. Results - Trees by Family

We can now use the properties of the spiral ordering from the previous section to construct a Gaussian prime labeling for several classes of trees. For each class of tree considered, we will give a definition, an example figure, and then provide our proof of the existence of a Gaussian prime labeling. We consider stars, paths, spiders, n-centipedes, (n, k, m)-double stars, $(n, 2)$-centipedes, and $(n, 3)$-firecrackers.

3.1. **Results on stars, paths, spiders, n-centipedes, and (n, k, m)-double stars.**

Definition 3.1. The **star graph**, S_n, on n vertices is the graph with

$$V(S_n) = \{v_1,\ v_2,\ \ldots,\ v_n\}, \quad \text{and} \quad E(S_n) = \{v_1 v_k : 2 \leq k \leq n\}.$$

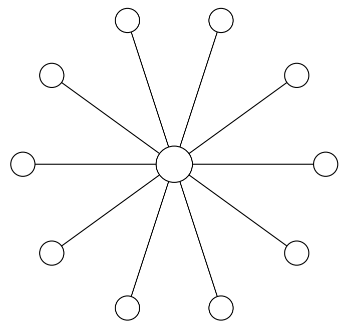

FIGURE 2. The star graph on 11 vertices

Theorem 3.2. *Any star graph admits a Gaussian prime labeling.*

Proof. Label vertex v_j with γ_j. The center vertex v_1 is then labeled with $\gamma_1 = 1$, which is relatively prime to all Gaussian integers. Therefore this is a Gaussian prime labeling. $\square$

Definition 3.3. The **path graph**, P_n, on n vertices is the graph with

$$V(P_n) = \{v_1,\ v_2,\ \ldots,\ v_n\} \quad \text{and} \quad E(P_n) = \{v_j v_{j+1} : 1 \leq j \leq n-1\}.$$

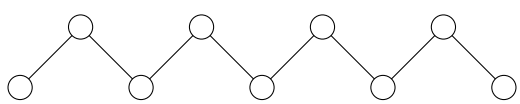

FIGURE 3. The path on 9 vertices

Theorem 3.4. *Any path admits a Gaussian prime labeling.*

Proof. Label vertex v_j with γ_j. By Corollary 2.13 consecutive Gaussian integers in the spiral ordering are relatively prime, so this is a prime labeling of the path. $\square$

Definition 3.5. A **spider graph** is a tree with one vertex of degree at least 3 and all other vertices having degree 1 or 2.

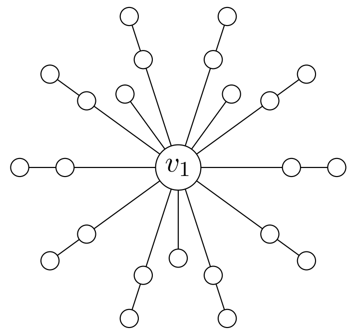

FIGURE 4. Example of a spider graph

Theorem 3.6. *Any spider tree admits a Gaussian prime labeling.*

Proof. Let T be a spider tree and suppose the center vertex v_1 has degree k. Then if we remove v_1 from T we are left with paths $L_1,\ L_2,\ \ldots,\ L_k$ with lengths $a_1,\ a_2,\ \ldots,\ a_k$ respectively. So label v_1 with 1 and label L_1 with the next a_1 consecutive Gaussian integers $\gamma_2,\ \gamma_3,\ \ldots,\ \gamma_{1+a_1}$, then label L_2 with the next a_2 consecutive Gaussian integers, and so on. By Corollary 2.13 this is a Gaussian prime labeling. □

Definition 3.7. The n**-centipede tree**, c_n, is the graph with

$$V(c_n) = \{v_1,\ v_2,\ \ldots,\ v_{2n}\},$$

and

$$E(c_n) = \{v_{2k-1}v_{2k} : 1 \leq k \leq n\} \cup \{\ v_{2k-1}v_{2k+1} : 1 \leq k \leq n-1\}.$$

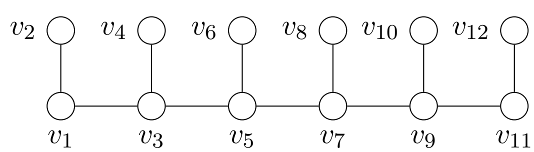

FIGURE 5. The 6-centipede tree

Theorem 3.8. *Any n-centipede tree admits a Gaussian prime labeling.*

Proof. Label vertex v_k with γ_k. This is a Gaussian prime labeling because consecutive Gaussian integers in the spiral ordering are relatively prime by Corollary 2.13 and consecutive odd Gaussian integers in the spiral ordering are relatively prime by Corollary 2.15. □

Definition 3.9. Let $n,\ k,\ m$ be integers with $k \leq m$. The (n, k, m)**-double star tree**, $DS_{n,k,m}$, is the graph with

$$V(DS_{n,k,m}) = \{v_1,\ \ldots,\ v_n,\ v_{n+1},\ \ldots,\ v_{n+k-1},\ v_{n+k},\ \ldots,\ v_{n+k+m-2}\},$$

and

$$E(DS_{n,k,m}) = \{v_jv_{j+1} : 1 \le j \le n-1, v_1v_{n+j} : 1 \le j \le k-1, v_nv_{n+k+j} : 0 \le j \le m-2\}.$$

In the (n,k,m)-double star tree we have a path of length n whose endvertices v_1 and v_n are the central vertices for stars on k and m vertices respectively (not including the other vertices on the path).

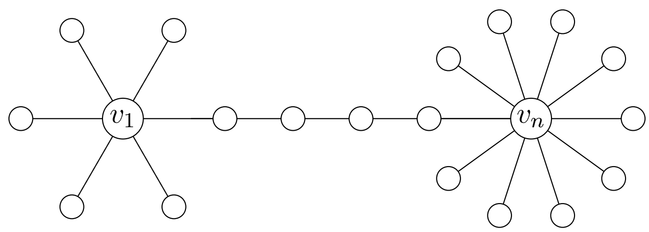

FIGURE 6. The $(6,6,11)$-double star tree

Theorem 3.10. *Any (n,k,m)-double star tree has a Gaussian prime labeling.*

Proof. Label v_n with $\gamma_1 = 1$, v_1 with $\gamma_2 = 1+\mathbf{i}$, and $v_2, \dots, v_{n-1}$ with the consecutive Gaussian integers $\gamma_3, \dots,\ \gamma_n$. We now label the $k-1$ remaining vertices adjacent to v_1 with odd Gaussian integers. If n is odd, label $v_{n+1}, \dots, v_{n+k-1}$ with $\gamma_{n+2},\ \gamma_{n+4}, \dots,\ \gamma_{n+2k-2}$. If n is even, label $v_{n+1}, \dots, v_{n+k-1}$ with $\gamma_{n+1},\ \gamma_{n+3}, \dots,\ \gamma_{n+2k-3}$. Label the remaining vertices adjacent to v_n arbitrarily with the remaining Gaussian integers in $[\gamma_{n+k+m-2}]$. This is a Gaussian prime labeling because$1+\mathbf{i}$ is relatively prime to all odd Gaussian integers, 1 is relatively prime to all Gaussian integers, and the path is labeled with consecutive Gaussian integers. □

Remark: When $n = 1$, this is a star graph and when $k = m = 0$ it is a path. When $k = m$ and $n = 2$, it is a firecracker graph.

3.2. **Results on $(n,2)$-centipede trees.**

Definition 3.11. The $(n,2)$**-centipede tree**, $c_{n,2}$, is the graph with

$$V(c_{n,2}) = \{v_1,\ v_2,\ \dots,\ v_{3n}\},$$

and

$$\begin{aligned} E(c_{n,2}) = &\{v_{3k-1}v_{3k-2},\ v_{3k-1}v_{3k} : 1 \le k \le n\} \cup \\ &\{v_{3k-1}v_{3k+2} : 1 \le k \le n-1\}. \end{aligned}$$

The $(n,2)$-centipede tree has n vertices on its spine with indices that are congruent to 1 mod 3. Each vertex on the spine has two leaf nodes adjacent to it. We call each spine vertex v_{3k-1} and its leaves v_{3k-2} and v_{3k} the k^{th} *segment* of the tree.

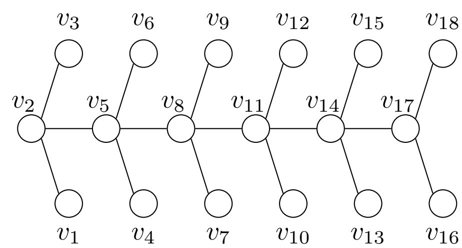

FIGURE 7. The (6,2)-centipede tree

Before we prove a result about $(n, 2)$-centipede trees, we will prove a lemma about Gaussian integers whose indices are three places apart in the spiral ordering.

Lemma 3.12. *Let $k \in \mathbb{Z}$ with $k \geq 2$, and consider the Gaussian integers γ_{3k-1} and γ_{3k+2}. Then $\delta = \gamma_{3k+2} - \gamma_{3k-1} \in \{\pm 3,\ \pm 3\mathbf{i},\ 1,\ \mathbf{i},\ 2-\mathbf{i},\ -2+\mathbf{i},\ 1-2\mathbf{i},\ -1+2\mathbf{i}\}$. Further, each of these values of δ does arise for some k.*

Proof. In the spiral ordering, Gaussian integers whose indices differ by 3 are three units apart. The possible combinations of three units are $3,\ 1,\ 1+2\mathbf{i},\ 2+\mathbf{i}$, and their associates. First, we know that the orientation of the spiral ordering will rule out $-2-\mathbf{i}, -1-2\mathbf{i}, -1$, and $-\mathbf{i}$. If γ_{3k-1} and γ_{3k+2} lie on the same branch, then they differ by ± 3 or $\pm 3\mathbf{i}$. Otherwise there is a corner between them and we consider three possibilities: γ_{3k-1} and γ_{3k+2} are separated by a corner on the real axis, γ_{3k-1} and γ_{3k+2} are separated by a corner on the line $\mathrm{Re}(z) = 1$, or γ_{3k-1} and γ_{3k+2} are separated by an interior corner.

Around corners on the real axis, we know γ_{3k-1} is of the form $a + 0\mathbf{i}$, $a + \mathbf{i}$, or $a + 2\mathbf{i}$ for some even integer a. If $\gamma_{3k-1} = a + 0\mathbf{i}$, by Theorem 2.11, we know its index is a^2 which is not congruent to $(3k-1) \equiv 2 \bmod 3$. Therefore $\gamma_{3k-1} \neq a + 0\mathbf{i}$. If $\gamma_{3k-1} = a + \mathbf{i}$, then by Theorem 2.11 we know that $3k - 1 = a^2 - 1$ which is possible when $a \equiv 0 \bmod 3$. In this case $\gamma_{3k+2} = (a+1) + \mathbf{i}$, so $\delta = 1$. If $\gamma_{3k-1} = a + 2\mathbf{i}$, by Theorem 2.11, we know that $3k - 1 = a^2 - 2$ which is possible when $a^2 \equiv 1 \bmod 3$. In this case $\gamma_{3k+2} = (a+1) + 0\mathbf{i}$, so $\delta = 1 - 2\mathbf{i}$. The possible values of δ for corners on the real axis are thus 1 and $1 - 2\mathbf{i}$.

Around corners on the line $\mathrm{Re}(z) = 1$, we know that γ_{3k-1} is of the form $1 + b\mathbf{i}, 2 + b\mathbf{i}$, or $3 + b\mathbf{i}$ for some even integer b or else γ_{3k-1} and γ_{3k+2} would lie on the same branch. If $\gamma_{3k-1} = 1 + b\mathbf{i}$, by Theorem 2.11 its index would be $(b+1)^2$, which is not congruent to $(3k-1) \equiv 2 \bmod 3$. Therefore $\gamma_{3k-1} \neq 1 + b\mathbf{i}$. If $\gamma_{3k-1} = 2 + b\mathbf{i}$, then Theorem 2.11 says that $3k - 1 = (b+1)^2 - 1$, which is possible when $b \equiv 2 \bmod 3$. In this case $\gamma_{3k+2} = 2 + (b + \mathbf{i})$ and so $\delta = \mathbf{i}$. If $\gamma_{3k-1} = 3 + b\mathbf{i}$, then Theorem 2.11 says that $3k - 1 = (b+1)^2 - 2$, which is possible when $b \equiv 0 \bmod 3$. In this case $\gamma_{3k+2} = 1 + (b + \mathbf{i})$ and so $\delta = -2 + \mathbf{i}$. The possible values of δ for corners on the line $\mathrm{Re}(z) = 1$ are thus $\mathbf{i}$ and $-2 + \mathbf{i}$.

Around interior corners there are two possible orientations. Either the spiral is moving from an up-oriented branch to a left-oriented branch or the spiral is moving from a right-oriented branch to a down-oriented branch. Consider the up-left interior corner. We know γ_{3k-1} has the form $a+(a-2)\mathbf{i}$ or $a+(a-3)\mathbf{i}$ as γ_{3k-1} and γ_{3k+2} would lie on the same branch otherwise. If $\gamma_{3k-1} = a+(a-2)\mathbf{i}$, Theorem 2.11 says that $3k-1 = (a-1)^2+1+a-2 = (a-1)^2+(a-1)$ which is possible when $a \equiv 1 \bmod 3$. In this case $\gamma_{3k+2} = (a-2)+(a-1)\mathbf{i}$ and so $\delta = -2+\mathbf{i}$. If $\gamma_{3k-1} = a+(a-3)\mathbf{i}$, Theorem 2.11 says that $3k-1 = (a-1)^2+1+a-3 = (a-1)^2+(a-1)-1$ which is possible when $a \equiv 0 \bmod 3$ or $a \equiv 1 \bmod 3$. In this case $\gamma_{3k+2} = (a-1)+(a-1)\mathbf{i}$ and so $\delta = -1+2\mathbf{i}$. By symmetry, the possible δ values for a right-down interior corner are $2-\mathbf{i}$ and $1-2\mathbf{i}$. The possible values of δ for interior corners are thus $-2+\mathbf{i},\ 2-\mathbf{i},\ -1+2\mathbf{i}$, and $1-2\mathbf{i}$. □

Theorem 3.13. *Any $(n,2)$-centipede tree admits a Gaussian prime labeling.*

Proof. Begin by labeling vertex v_j with γ_j. We call the set of nodes $\{v_{3k-1}\}_{k=1}^{n}$ which have degree greater than one the spine of the tree, and the other nodes the leaves. Each spine-leaf pair in the same segment will be relatively prime because they are labeled by consecutive Gaussian integers, but there will be adjacent nodes down the spine that are not relatively prime.

Consider the possible separations of pairs of nodes on the spine. Let $\delta_k = \gamma_{3k+2} - \gamma_{3k-1}$. By Lemma 3.12, $\delta_k \in \{\pm 3,\ \pm 3\mathbf{i},\ 1,\ \mathbf{i},\ 2-\mathbf{i},\ -2+\mathbf{i},\ 1-2\mathbf{i},\ -1+2\mathbf{i}\}$. This presents a problem whenever δ_k is not a unit, but it divides both γ_{3k+2} and γ_{3k-1}. To solve this problem, we will swap the labels of some spine nodes with the labels of one of their leaf neighbors.

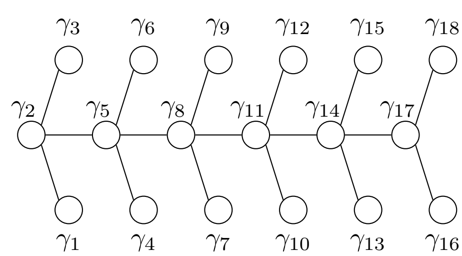

Figure 8. Initial labeling of the (6,2)-centipede

Swap each even γ_{3k-1} with γ_{3k} and consider the new labeling. In each segment of the caterpillar where a swap occurred, the node on the spine will still be relatively prime to its leaves, as one is consecutive to it and the other is a consecutive odd to it.

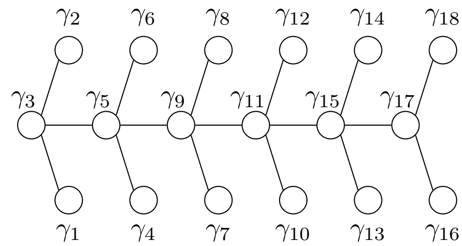

FIGURE 9. Labeling of the (6,2)-centipede after the swap

Now we verify that the labels of the nodes along the spine are relatively prime. Along the spine, we will have a sequence of odd Gaussian integers with indices $3, 5, 9, 11, \ldots, 6k+3, 6k+5, \ldots$. Every other pair of Gaussian integers in this sequence (γ_3, γ_5 for example) is a pair of consecutive odds and by Corollary 2.15 is relatively prime. The other pairs (γ_5, γ_9 for example) are odds that are four indices apart. We claim that in this labeling these labels will always be relatively prime.

First, let $k > 1$ be odd so that we are considering a segment with an even node on the spine and consider $\delta_k := \gamma_{3k-1} - \gamma_{3k-4}$ and $\delta'_k := \gamma_{3k} - \gamma_{3k-4}$. Note that choosing k to be odd restricts the possibilities from Lemma 3.12 further because of the parities of types of corners. Now $\delta_k \in \{\pm 3,\ \pm 3\mathbf{i},\ 1,\ 2-\mathbf{i},\ -2+\mathbf{i},\ 1-2\mathbf{i}\}$. Because γ_{3k-1} is even and has an index congruent to 2 mod 3 we also know that γ_{3k-1} either does not lie on a corner in the spiral ordering or lies on an even corner on the line $\mathrm{Re}(z) = 1$.

If δ_k is equal to $3, -3,\ 3\mathbf{i}$, or $-3\mathbf{i}$ then γ_{3k-1} and γ_{3k} lie on the same branch. Thus $\gamma_{3k} - \gamma_{3k-1} = 1, -1,\ \mathbf{i}$, or $-\mathbf{i}$ respectively. Therefore $\delta'_k = 4, -4, 4\mathbf{i}$, or $-4\mathbf{i}$ respectively. If $\delta_k = 1$, then $\delta'_k = 1+\mathbf{i}$. This occurs around corners on the real axis. If $\delta_k = 2 - \mathbf{i}$, then $\delta'_k = 2 - 2\mathbf{i}$ because this occurs around interior corners of the spiral. If $\delta_k = -2+\mathbf{i}$ then $\delta'_k = -1+\mathbf{i}$ because this occurs around $\mathrm{Re}(z) = 1$ corners. If $\delta_k = 1 - 2\mathbf{i}$, then $\delta'_k = 1 - \mathbf{i}$. All of these δ'_k are powers of $1+\mathbf{i}$, so by Lemma 2.14 the odds along the spine are now relatively prime. Thus this is a prime labeling of the $(n, 2)$-centipede. □

3.3. **Results on $(n, 3)$-firecracker trees.** The proof that $(n, 2)$-caterpillar trees admit a Gaussian prime labeling relied on an initial natural labeling that was slightly modified to give a prime labeling. In this section we use a similar technique to show that certain firecracker trees also admit prime labelings.

Definition 3.14. The $(n, 3)$**-firecracker tree**, $F_{n,3}$ is the graph with

$$V(F_{n,3}) = \{v_1,\ v_2,\ \ldots,\ v_{3n}\},$$

and

$$E(F_{n,3}) = \{v_{3k-2}v_{3k-1},\ v_{3k-1}v_{3k} : 1 \le k \le n\} \cup \{\ v_{3k-2}v_{3k+1} : 1 \le k \le n-1\}.$$

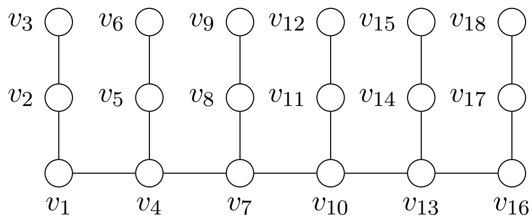

FIGURE 10. The $(6,3)$-firecracker tree

We call the n vertices $v_1,\ v_4,\ v_7, \ldots,\ v_{3n-2}$ the spine of the tree. The set of vertices $v_{3k-2},\ v_{3k-1},\ v_{3k}$ for some k is called the k^{th} level of the tree.

Lemma 3.15. *Let $\alpha = a + b\mathbf{i}$ be a Gaussian integer. Then $1 + 2\mathbf{i} \mid \alpha$ if and only if $5 \mid a + 2b$ and $2 + \mathbf{i} \mid \alpha$ if and only if $5 \mid 2a + b$.*

Proof. Suppose $1 + 2\mathbf{i} \mid \alpha$. Then $\frac{a+b\mathbf{i}}{1+2\mathbf{i}} = \frac{(a+b\mathbf{i})\cdot(1-2\mathbf{i})}{(1+2\mathbf{i})\cdot(1-2\mathbf{i})} = \frac{(a+2b)+(b-2a)\mathbf{i}}{5} \in \mathbb{Z}[\mathbf{i}]$. Hence $5 \mid a + 2b$. Conversely, suppose $5 \mid a + 2b$ such that $a + 2b = 5m$. It follows that $a = 5m - 2b$ so $b - 2a = b - 2(5m - 2b) = 5b - 10m$, and $5 \mid b - 2a$. Therefore $5 \mid (a + 2b) + (b - 2a)\mathbf{i}$, so $5 \mid (a + b\mathbf{i})(1 - 2\mathbf{i})$, and $(1 + 2\mathbf{i})(1 - 2\mathbf{i}) \mid (a + b\mathbf{i})(1 - 2\mathbf{i})$. Hence $1 + 2\mathbf{i} \mid \alpha$.

The argument for $2 + \mathbf{i}$ is similar. □

This lemma is illustrated in Table 1, which shows the mod 5 residues of a multiple of $1 + 2\mathbf{i}$ or $2 + \mathbf{i}$.

Re(α)	Im(α)
0	0
1	2
2	4
3	1
4	3

(A) $1 + 2\mathbf{i} \mid \alpha$

Re(α)	Im(α)
0	0
1	3
2	1
3	4
4	2

(B) $2 + \mathbf{i} \mid \alpha$

TABLE 1. Components of multiples of $1 + 2\mathbf{i}$ and $2 + \mathbf{i}$ reduced mod 5

Lemma 3.16. *For $k \in \mathbb{N}$, let $\delta_k = \gamma_{3k+1} - \gamma_{3k-2}$ and assume that δ_k is not a unit. If $\delta_k \mid \gamma_{3k-2}$ and $\delta_k \mid \gamma_{3k+1}$ then one of the following six conditions holds:*

(1) $\delta_k = -1 + 2\mathbf{i}$ and $\gamma_{3k-2} = a + (a - 3)\mathbf{i}$ for $a \equiv 1 \bmod 5$ and a odd,
(2) $\delta_k = 2 - \mathbf{i}$ and $\gamma_{3k-2} = (a - 2) + (a - 1)\mathbf{i}$ for $a \equiv 3 \bmod 5$ and a even,
(3) $\delta_k = 1 - 2\mathbf{i}$ and $\gamma_{3k-2} = (a - 1) + 2\mathbf{i}$ for $a \equiv 0 \bmod 5$ and a odd,
(4) $\delta_k = 1 + 2\mathbf{i}$ and $\gamma_{3k-2} = a + 0\mathbf{i}$ for $a \equiv 0 \bmod 5$ and a even,
(5) $\delta_k = -2 + \mathbf{i}$ and $\gamma_{3k-2} = 3 + b\mathbf{i}$ for $b \equiv 1 \bmod 5$ and b even,

(6) $\delta_k = 2 + \mathbf{i}$ *and* $\gamma_{3k-2} = 1 + b\mathbf{i}$ *for* $b \equiv 3 \bmod 5$ *and* b *even.*

Proof. First, by Lemma 2.17 δ_k is not equal to 3 or one of its associates. Because γ_{3k-2} and γ_{3k+1} are three indices apart in the spiral ordering and δ_k is not $1, 3$, or any of their associates, δ_k must be $1+2\mathbf{i}$, $2+\mathbf{i}$, or one of their associates. We also know that $\delta_k \neq -1-2\mathbf{i}, -2-\mathbf{i}$ by the orientation of the spiral ordering. Now we consider the remaining associates according to the location of γ_{3k-2} in the spiral ordering.

Case 1*: Interior corners on up-left branches.* This case corresponds to $\delta_k = -1+2\mathbf{i}$ or $\delta_k = -2+\mathbf{i}$. If $\delta_k = -2+\mathbf{i}$, then $\gamma_{3k-2} = a + (a-2)\mathbf{i}$ for some a because it is one step below the interior corner $a + (a-1)\mathbf{i}$. Then Theorem 2.11 says that its index is $(a-1)^2 + (a-2) + 1$, which is always congruent to 0 or 2 mod 3. This contradicts that the index is $3k - 2 \equiv 1 \bmod 3$. Therefore this δ_k does not occur.

If $\delta_k = -1 + 2\mathbf{i} = \mathbf{i}(2 + \mathbf{i})$, then by Theorem 2.11 $\gamma_{3k-2} = a + (a-3)\mathbf{i}$ for some odd a because it is two steps below the interior corner $a + (a-1)\mathbf{i}$. Using Table 1 we see that we must have $a \equiv 1 \bmod 5$ for this to occur.

Case 2*: Interior corners on right-down branches.* This case corresponds to $\delta_k = 2 - \mathbf{i}$ or $\delta_k = 1 - 2\mathbf{i}$. If $\delta_k = 1 - 2\mathbf{i}$, then $\gamma_{3k-2} = (a-1) + (a-1)\mathbf{i}$ for some a because it is one step left of the interior corner $a + (a-1)\mathbf{i}$. Theorem 2.11 says that its index is $(a-1)^2 + (a-1)$, which is always congruent to 0 or 2 mod 3. Again this contradicts that the index is $3k - 2 \equiv 1 \bmod 3$. So this δ_k does not occur.

If $\delta_k = 2 - \mathbf{i} = -\mathbf{i}(1 + 2\mathbf{i})$, then by Theorem 2.11 $\gamma_{3k-2} = (a-2) + (a-1)\mathbf{i}$ for some even a because it is two steps left of the interior corner $a + (a-1)\mathbf{i}$. Using Table 1 we see that we must have $a \equiv 3 \bmod 5$ for this to occur.

Case 3*: Corners on the real axis.* This case corresponds to $\delta_k = 1 - 2\mathbf{i}$ or $\delta_k = 1 + 2\mathbf{i}$. If $\delta_k = 1 - 2\mathbf{i} = -\mathbf{i}(2 + \mathbf{i})$ then $\gamma_{3k+1} = a + 0\mathbf{i}$ and $\gamma_{3k-2} = (a-1) + 2\mathbf{i}$ for some odd a (by Theorem 2.11) with $a \equiv 0 \bmod 5$ (by Table 1).

If $\delta = 1 + 2\mathbf{i}$, then $\gamma_{3k-2} = a + 0\mathbf{i}$ and $\gamma_{3k+1} = (a+1) + 2\mathbf{i}$ for some even a (by Theorem 2.11) with $a \equiv 0 \bmod 5$ (by Table 1).

Case 4*: Corners on the line* $\mathrm{Re}(z) = 1$. This case corresponds to $\delta_k = 2 + \mathbf{i}$ or $\delta_k = -2 + \mathbf{i}$. If $\delta_k = -2 + \mathbf{i} = \mathbf{i}(1 + 2\mathbf{i})$, then by Theorem 2.11 $\gamma_{3k-2} = 3 + b\mathbf{i}$ for some even b because it is 2 away from the corner on the $\mathrm{Re}(z) = 1$ line at $1 + b\mathbf{i}$. Using Table 1 we see that we must have $b \equiv 1 \bmod 5$ for this to occur.

If $\delta_k = 2 + \mathbf{i}$, then by Theorem 2.11 $\gamma_{3k-2} = 1 + b\mathbf{i}$ for some even b because it is a corner on the $\mathrm{Re}(z) = 1$ line. Using Table 1 we see that we must have $b \equiv 3 \bmod 5$ for this to occur. □

Theorem 3.17. *Any* $(n, 3)$*-firecracker tree has a Gaussian prime labeling.*

Proof. A natural first attempt at labeling the $(n, 3)$-firecracker is to label it consecutively by labeling v_j with γ_j for all j. This is a nearly prime labeling, so we make a handful of swaps of labels to resolve the issues that appear and give a fully prime labeling. The full labeling procedure is as follows.

We define an initial labeling ℓ_1 of the $(n,3)$-firecracker so that $\ell_1(v_j) = \gamma_j$. Now we define a new labeling ℓ in the following way. Let $\delta_k = \ell_1(v_{3k+1}) - \ell_1(v_{3k-2})$. Consider each k where δ_k is not a unit, $\delta_k \mid \ell_1(v_{3k-2})$, and $\delta_k \mid \ell_1(v_{3k+1})$. By Lemma 3.16 we know that $\delta_k \in \{-1+2\mathbf{i},\ 2-\mathbf{i},\ 1-2\mathbf{i},\ 1+2\mathbf{i},\ -2+\mathbf{i},\ 2+\mathbf{i}\}$. If $\delta_k = -1+2\mathbf{i},\ 2-\mathbf{i},\ 1+2\mathbf{i}$, or $-2+\mathbf{i}$, then we set $\ell(v_{3k-2}) = \gamma_{3k}$ and $\ell(v_{3k}) = \gamma_{3k-2}$. If $\delta_k = 1-2\mathbf{i}$, then we set $\ell(v_{3k+1}) = \gamma_{3k+3}$ and $\ell(v_{3k+3}) = \gamma_{3k+1}$. If $\delta_k = 2+\mathbf{i}$, then we set $\ell(v_{3k-2}) = \gamma_{3k}$, $\ell(v_{3k}) = \gamma_{3k-2}$, $\ell(v_{3k-5}) = \gamma_{3k-3}$, and $\ell(v_{3k-3}) = \gamma_{3k-5}$. For all other v_j, keep $\ell(v_j) = \ell_1(v_j) = \gamma_j$.

Because the effect of each above change is to swap the labels of vertices at the ends of a particular level, adjacent vertices on the same level are still labeled with consecutive Gaussian integers in the spiral ordering, which are relatively prime. Therefore, we only need to show that this new labeling has created a prime labeling on the spine. To show the vertices along the spine are now relatively prime, we examine each edge along the spine. If neither vertex on a given edge is affected by the relabeling, then the labels are relatively prime. Otherwise we must inspect each relabeled vertex to make sure its new label is relatively prime to its neighbors on the spine. To do this, we consider each δ_k in turn. Note that the conditions from Lemma 3.16 force k to be large enough that if $\delta_k \mid \gamma_{3k+1}$ and $\delta_k \mid \gamma_{3k-2}$ then γ_{3k+1} and γ_{3k+3} are on the same branch and γ_{3k-2} and γ_{3k-5} are on the same branch and so neither γ_{3k+3} or γ_{3k-5} was relabeled.

If $\delta_k = -1+2\mathbf{i}$, then $\ell(v_{3k-2}) = \gamma_{3k}$. By Lemma 3.16 γ_{3k} lies at an interior corner $a+(a-1)\mathbf{i}$ with $a \equiv 1 \bmod 5$. Since $\ell(v_{3k-2}) = \gamma_{3k}$ and $\ell(v_{3k+1}) = \gamma_{3k+1}$ are relatively prime, we need only check whether $\ell(v_{3k-2})$ and $\ell(v_{3k-5})$ are relatively prime. Then $\delta' := \ell(v_{3k-2}) - \ell(v_{3k-5}) = \gamma_{3k} - \gamma_{3k-5} = 5\mathbf{i}$, so their only possible common factors are $1+2\mathbf{i}$ and $2+\mathbf{i}$. Consulting Table 1, we see that because $\mathrm{Re}(\gamma_{3k}) \equiv 1 \bmod 5$ and $\mathrm{Im}(\gamma_{3k}) \equiv 0 \bmod 5$ neither $1+2\mathbf{i}$ or $2+\mathbf{i}$ divides γ_{3k} and so $\ell(v_{3k-2})$ and $\ell(v_{3k-5})$ are relatively prime.

If $\delta_k = 2-\mathbf{i}$, then $\ell(v_{3k-2}) = \gamma_{3k}$. Note that γ_{3k} lies at an interior corner $a+(a-1)\mathbf{i}$ for some $a \equiv 3 \bmod 5$ by Lemma 3.16. Here $\delta' := \ell(v_{3k-2}) - \ell(v_{3k-5}) = \gamma_{3k} - \gamma_{3k-5} = 5$, so the only possible common factors are again $1+2\mathbf{i}$ and $2+\mathbf{i}$. Consulting Table 1, we see that because $\mathrm{Re}(\gamma_{3k}) \equiv 3 \bmod 5$ and $\mathrm{Im}(\gamma_{3k}) \equiv 2 \bmod 5$ neither $2+\mathbf{i}$ nor $1+2\mathbf{i}$ divides γ_{3k} and thus $\ell(v_{3k-2})$ and $\ell(v_{3k-5})$ are relatively prime.

If $\delta_k = 1-2\mathbf{i}$ then $\gamma_{3k+1} = a+0\mathbf{i}$ for some odd $a \equiv 0 \bmod 5$ and $\ell(v_{3k+1}) = \gamma_{3k+3}$. Since $\ell(v_{3k+1}) = \gamma_{3k+3}$ and $\ell(v_{3k+4}) = \gamma_{3k+4}$ are consecutive, they are relatively prime and we need only consider the new $\delta' := \ell(v_{3k+1}) - \ell(v_{3k-2}) = \gamma_{3k+3} - \gamma_{3k-2} = (a+2\mathbf{i}) - ((a-1)+2\mathbf{i}) = 1$. So $\ell(v_{3k-2})$ and $\ell(v_{3k+1})$ are also relatively prime.

If $\delta_k = 1+2\mathbf{i}$, then $\gamma_{3k-2} = a+0\mathbf{i}$ for some even $a \equiv 0 \bmod 5$ and $\ell(v_{3k-2}) = \gamma_{3k}$. Since $\ell(v_{3k-2}) = \gamma_{3k}$ and $\ell(v_{3k+1}) = \gamma_{3k+1}$ are consecutive, they are relatively prime and we only consider the new $\delta' := \ell(v_{3k-2}) - \ell(v_{3k-5}) = \gamma_{3k} - \gamma_{3k-5} = 1 - 2\mathbf{i}$. Because γ_{3k-2} is a multiple of both $1+2\mathbf{i}$ and $2+\mathbf{i}$, it follows that γ_{3k} is a multiple of neither. So $\ell(v_{3k-2})$ and $\ell(v_{3k-5})$ are relatively prime.

If $\delta_k = -2+\mathbf{i}$, then $\ell(v_{3k-2}) = \gamma_{3k} = 1 + b\mathbf{i}$ for some $b \equiv 1 \bmod 5$. Since $\ell(v_{3k+1}) = \gamma_{3k+1}$ and $\ell(v_{3k-2}) = \gamma_{3k}$ are consecutive, they are relatively prime and we need only look at $\delta' := \ell(v_{3k-2}) - \ell(v_{3k-5}) = \gamma_{3k} - \gamma_{3k-5}$. Consulting Table 1, we see that because $\mathrm{Re}(\gamma_{3k}) = 1$ and $\mathrm{Im}(\gamma_{3k}) \equiv 1 \bmod 5$ neither $1+2\mathbf{i}$ or $2+\mathbf{i}$ divides γ_{3k}. So $\ell(v_{3k-2})$ and $\ell(v_{3k-5})$ are relatively prime.

If $\delta_k = 2+\mathbf{i}$, then $\ell(v_{3k-2}) = \gamma_{3k} = 2 + (b+1)\mathbf{i}$ for some $b \equiv 3 \bmod 5$. We also have $\ell(v_{3k+1}) = \gamma_{3k+1} = 3 + (b+1)\mathbf{i}, \ell(v_{3k-8}) = \gamma_{3k-8} = 7 + b\mathbf{i}$, and $\ell(v_{3k-5}) = \gamma_{3k-3} = 2 + b\mathbf{i}$. We now have the sequence of labels on the spine of γ_{3k-8}, γ_{3k-3}, γ_{3k}, γ_{3k+1}. First, γ_{3k} and γ_{3k+1} are consecutive and thus relatively prime. Also, γ_{3k-3} and γ_{3k} have a difference of $\mathbf{i}$ and are thus relatively prime. Finally, $\delta' := \ell(v_{3k-5}) - \ell(v_{3k-8}) = \gamma_{3k-3} - \gamma_{3k-8} = -5$. Consulting Table 1 we see that because $\mathrm{Re}(\gamma_{3k-3}) = 2$ and $\mathrm{Im}(\gamma_{3k-3}) \equiv 3 \bmod 5$ neither $1+2\mathbf{i}$ or $2+\mathbf{i}$ divides γ_{3k-3}, so γ_{3k-3} and γ_{3k-8} are also relatively prime.

Thus ℓ is a prime labeling of the $(n,3)$-firecracker tree. $\square$

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