

## The Master Method

Let  $a \geq 1$  and  $b > 1$  be integers, and let  $f(n)$  be a non-negative function. The master method provides solutions for many recursions of the form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

(with any constant-time base case).

**Case 1** : if  $f(n) = O(n^{\log_b a - \epsilon})$  for some constant  $\epsilon > 0$ , then

$$T(n) = \Theta(n^{\log_b a}).$$

**Case 2** : if  $f(n) = \Theta(n^{\log_b a} \log^k n)$ , then

$$T(n) = \Theta(n^{\log_b a} \log^{k+1} n).$$

**Case 3** : if  $f(n) = \Omega(n^{\log_b a + \epsilon})$  for some constant  $\epsilon > 0$ , and “ $f$  is not strange,” then

$$T(n) = \Theta(f(n)).$$

By “ $f$  is not strange,” we mean that  $f$  satisfies the following *regularity condition* with respect to the  $a$  and  $b$  appearing in the recurrence: for some constant  $c < 1$  and all sufficiently large  $n$ ,

$$af(n/b) \leq cf(n).$$

This condition is rarely violated in practice. In particular, it is satisfied for any  $f(n)$  that is a polynomial or looks like  $cn^j \log^k n$  for any  $c, j, k \geq 0$ .

In class, I claimed that polynomials are not strange. Here's a more precise statement with the proof:

**Lemma:** *Suppose that we have a recurrence of the form*

$$T(n) = aT\left(\frac{n}{b}\right) + n^j$$

*to which Case 3 of the master method might apply; that is,  $j = \log_b a + \epsilon$  for some constant  $\epsilon > 0$ . Then*

$$a\left(\frac{n}{b}\right)^j = \frac{1}{b^\epsilon} n^j.$$

Since  $b > 1$ , this inequality proves that  $n^j$  does not violate the regularity condition.

**Proof:**

$$\begin{aligned} a\left(\frac{n}{b}\right)^j &= \frac{an^j}{b^j} \\ &= \frac{an^j}{b^{(\log_b a + \epsilon)}} \\ &= \frac{an^j}{b^{\log_b a} \cdot b^\epsilon} \\ &= \frac{an^j}{ab^\epsilon} \\ &= \frac{n^j}{b^\epsilon} \quad \text{QED.} \end{aligned}$$

Technically, this proof only covers monomials, but it's not too hard to show that the sum or difference of two monomials isn't strange either (provided that the latter is non-negative, of course), and we can then expand non-strangeness inductively to arbitrary polynomials.

It's also not hard to show that multiplying  $n^j$  by  $\log^k n$  for any  $k$  doesn't cause it to become strange:

**Lemma:** *Suppose that we have a recurrence of the form*

$$T(n) = aT\left(\frac{n}{b}\right) + n^j \log^k n$$

*to which Case 3 of the master method might apply; that is,  $k = \log_b a + \epsilon$  for some constant  $\epsilon > 0$ . Then*

$$a\left(\frac{n}{b}\right)^j \log^k\left(\frac{n}{b}\right) \leq \frac{1}{b^\epsilon} n^j \log^k n.$$

**Proof:**

$$\begin{aligned} a\left(\frac{n}{b}\right)^j \log^k\left(\frac{n}{b}\right) &= \frac{an^j}{b^j} \log^k\left(\frac{n}{b}\right) \\ &= \frac{n^j}{b^\epsilon} (\log n - \log b)^k \\ &\leq \frac{1}{b^\epsilon} n^j \log^k n \quad \text{QED.} \end{aligned}$$

Hence, it's rather difficult to find recurrences with strange functions in the practical analysis of algorithms.

Just to show you that there *are* some strange functions out there, consider the recurrence

$$T(n) = T\left(\frac{n}{2}\right) + 4^{\lceil \log_4 n \rceil}.$$

For this recurrence,  $a = 1$  and  $b = 2$ , and it's not hard to see that  $4^{\lceil \log_4 n \rceil} \geq n$ , which is polynomially greater than  $n^{\log_2 1} = 1$ . However,  $4^{\lceil \log_4(n/2) \rceil} = 4^{\lceil \log_4 n \rceil}$  whenever  $n$  is a power of 4, so the regularity condition of Case 3 is violated.