RESEARCH STATEMENT

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1. Overview

Recent months have seen major progress toward completing the Minimal Model Program (MMP), bringing us tantalizingly close to a method for classifying algebraic varieties; however, several holes remain to fill before the problem is solved in its full generality. The primary goal of the MMP is fairly simple to state – given a variety Y, construct a variety X birational to Y with the property that the canonical divisor K_X is *numerically effective*, or nef: the intersection of K_X with a curve is always non-negative. Such a variety is called a *minimal model* for Y. Various classes of singularities have been constructed in the effort to complete the MMP, allowing for several types of models – for example, the definition of *canonical models* has the additional condition that X has only canonical singularities.

With this in mind, we can define *semi log canonical models*. As in the case of canonical models, this introduces the condition that X has *semi log canonical singularities* (another class introduced in connection with the MMP; it has the benefit of not requiring a variety to be normal). Focusing our attention on smooth projective varieties, we consider the following conjecture, which hints at a possible method of constructing "nice" semi log canonical models.

Conjecture. *Let Y be a smooth projective variety of dimension n, suitably embedded in some projective space. Then the generic linear projection of Y to a hypersurface in $\mathbb{P}^{n+1}$ is semi log canonical.*

If this conjecture were to hold (for an appropriate definition of "suitably embedded"), then it would give some evidence for the possibility that we could construct semi log canonical models which are actually *hypersurfaces*. The only missing piece would be a method for ensuring K_X is nef, perhaps via a re-embedding of Y.

Applying some new results about Du Bois singularities to known results about generic projections, I have successfully shown that hypersurfaces as above are semi log canonical when $\dim Y \leq 5$. (In the course of my work, I have proven a connection between certain Du Bois singularities and semi log canonical ones.) Unfortunately, discussion of my work with others has turned up a class of examples where the conjecture fails – at least when $\dim Y \geq 30$. However, all is not lost: it turns out that this class of examples fails precisely because its members are already minimal models!

At this point, several possible directions of inquiry arise. One natural statement to hypothesize is that, if the smooth variety Y is not a minimal model, then the image hypersurface X is indeed semi log canonical. If this is true, then it would still remain to determine whether it's possible to force K_X to be nef. An interesting alternative approach has been suggested as well – rather than considering only generic linear projections, it may be worth considering generic points of the hom scheme $\mathrm{Hom}(X, \mathbb{P}^{n+1})$.

2. Statement of Main Results

In order to properly state the rigorous version of the results mentioned above, we first give some definitions.

For any scheme X over $\mathbb{C}$ there exists a complex $\underline{\Omega}^{\cdot}_X \in D_F(X)$ satisfying various useful properties, such as the fact that the analogue of the Hodge-de Rham spectral sequence exists. In general its construction is quite difficult, though for smooth schemes $\underline{\Omega}^{\cdot}_X \simeq_{\mathrm{qis}} \Omega^{\cdot}_X$, where $\Omega^{\cdot}_X$ is the usual de Rham complex with the *filtration bête* ($\simeq_{\mathrm{qis}}$ denotes quasi-isomorphism). We denote $\mathrm{Gr}^0_F \underline{\Omega}^{\cdot}_X$ by $\underline{\Omega}^0_X$, and we say X has Du Bois singularities if $\underline{\Omega}^0_X \simeq_{\mathrm{qis}} \mathscr{O}_X$. So in particular, it's clear that smooth schemes are Du Bois.

Let X be an S_2 scheme with only double normal crossing singularities in codimension 1, and with K_X $\mathbb{Q}$-Cartier. A *semiresolution* of X is a map $f : Y \to X$ such that Y is semismooth (i.e., every point of Y is either a double normal crossing point or a pinch point), and such that f is an isomorphism outside a codimension 2 subset. We call f a *good semiresolution* if $D_Y \cup E$ is a simple normal crossing divisor, where D_Y is the double locus of Y and E is the exceptional divisor of f. If f is a semiresolution, we can write

$$K_Y \equiv f^* K_X + \sum_i a_i E_i,$$

with $a_i \in \mathbb{Q}$. If f is a good semiresolution and for every i $a_i \geq -1$, then we say X is *semi log canonical* (slc).

With these definitions in mind, we state some new results concerning slc and Du Bois singularities.

Theorem 1. *Suppose X has double normal crossings in codimension 1, and K_X is Cartier. If X has Du Bois singularities, then X has slc singularities.*

Corollary 2. *Suppose X is Gorenstein and seminormal. If X has Du Bois singularities, the X has slc singularities.*

These statements demonstrate that Du Bois singularities can provide an excellent tool for detecting slc singularities (which is difficult in general). The following theorem provides an example of applying this technique.

Theorem 3. *Let Y be a smooth projective variety of dimension n, embedded in some $\mathbb{P}^N$ via a d-uple embedding with $d \geq 3n$. Let $f : Y \to X$ be a generic projection of Y to a hypersurface in $\mathbb{P}^{n+1}$. If $n \leq 5$, then X is semi log canonical.*

The proof of this theorem relies on applying the above corollary to some long-known results regarding generic projections. It is tempting to try to extend this statement to all dimensions, but the following shows this is impossible:

Theorem 4. *Let X and Y be as in the previous theorem. For any $n \geq 30$, there exists a Y such that X is not semi log canonical.*

Curiously, the counterexamples are all minimal models, and the proof of this theorem actually relies on that property.