The focus of my research is in the area of Integrable Systems. In its most basic sense, an integrable system is a system of differential equations that can be solved in some “explicit” form. The modern area of study in this field began in the late 1960’s and early 1970’s, when what is now often referred to as the scattering and inverse scattering transform was developed to study various integrable partial differential equations (PDE).

The scattering/inverse scattering transform can be thought of as a nonlinear analogue to Fourier analysis. As in Fourier analysis, an initial condition is mapped to spectral information. Due to the integrability, the evolution of this spectral data is then easily determined. The inverse scattering transform is the reconstruction of the solution to the governing PDE from the evolved spectral data. The inverse scattering method can be realized for many integrable PDEs via an associated Riemann-Hilbert problem (RHP). A Riemann-Hilbert problem consists of constructing an analytic function (possibly matrix valued) from its singularities. Typically the singularities are given in terms of jump conditions on contours as well as isolated singularities. In [12], Deift and Zhou developed a nonlinear steepest decent technique for oscillatory Riemann-Hilbert problems in order to completely describe the long time asymptotics of modified Korteweg de Vries (MKdV) equation. The basic techniques they used have been extended and utilized to prove many other results for integrable PDE. In particular, in [10] Deift and Zhou obtained large time asymptotics to the defocusing nonlinear Schrödinger equation (NLS) for a large class of initial data as a case study of this method.

In addition to the success of the Riemann-Hilbert method in analyzing integrable PDEs, several mathematically and physically important questions dealing with other integrable systems have been answered using this method. Most notably, many problems in Random Matrix Theory and Orthogonal Polynomials have been answered using these methods. The survey articles by Deift [7] and Its [18] explain some of the these techniques and indicate many of the results proved using the Riemann-Hilbert method.

My research thus far has dealt with problems dealing with integrable PDEs as well as questions in random matrix theory and orthogonal polynomials. In the remainder of this statement, I will describe three projects I have worked on. The first involves the forward scattering transform for the modified nonlinear Schrödinger equation (MNLS), where in collaboration with Peter Miller, we have proved several facts about the spectrum associated to this PDE. The second is a problem involving inverse scattering and Riemann-Hilbert analysis where, jointly with Kenneth T.-R. McLaughlin, we proved a nonlinear analogue of the Gibbs phenomenon for the defocusing nonlinear Schrödinger (NLS) equation. The third project involves random matrix theory and its connections to Riemann-Hilbert problems. Jinho Baik, Robert Buckingham and myself have calculated asymptotic facts for the Tracy-Widom distributions corresponding to the largest eigenvalue statistics for the Gaussian Unitary, Orthogonal, and Symplectic ensembles. I also indicate areas of ongoing research related to these problems as well as future directions.

**Semi-classical analysis of the modified nonlinear Schrödinger equation**

The modified nonlinear Schrödinger (MNLS) equation is

\[ i\varepsilon \frac{\partial \phi}{\partial t} + \frac{\varepsilon^2}{2} \frac{\partial^2 \phi}{\partial x^2} + |\phi|^2 \phi + i\varepsilon \alpha \frac{\partial}{\partial x} (|\phi|^2 \phi) = 0. \]

The correct problem to pose for this equation when \( x \in \mathbb{R} \) is the Cauchy problem where \( g(x) = \phi(x, t = 0) \) is a given initial condition. Together with Peter Miller, I have proved several facts (which will be outlined below) about the forward scattering transform associated to the MNLS equation and the resulting spectral data.

The focusing NLS equation is a special case of the MNLS equation when \( \alpha = 0 \). The focusing NLS is well known and arises naturally in the modeling of any nearly monochromatic, weakly nonlinear, dispersive wave propagation. In particular, it is (in many circumstances) an adequate model to describe the electric field envelope of picosecond pulses in monomode optical fibers [17]. The so-called “semiclassical”, or zero-dispersion limit consists of taking the initial condition in the form \( g(x) = A(x)e^{iS(x)/\varepsilon} \), for fixed real valued functions \( A(x) \) and \( S(x) \), and considering the behavior of solutions when \( \varepsilon \downarrow 0 \). This limiting regime is important in the design of (increasingly prevalent) dispersion-shifted fibers. The utility of the focusing NLS equation stems from the fact that it is an integrable PDE. However, if higher order effects become important, it is necessary to account for these effects in the model. It is then reasonable to ask if one can include higher order effects in the model and maintain the integrability. The MNLS does just this [16]. Additionally, for the same reasons as in the focusing NLS model, the semiclassical limit of the MNLS model is of particular interest in the design of dispersion-shifted fibers. Since it also has the property of being integrable, we can use the tools of the scattering/inverse scattering transform to analyze properties of solutions to the MNLS.
Since the focusing NLS equation is the special case of the MNLS equation \((\alpha = 0)\), understanding the MNLS for small \(\alpha\) could provide new information about the focusing NLS equation as well.

Upon initial investigation, we discovered that the MNLS has the interesting property that, for certain initial conditions, the system is modulationally stable. Consequently, for initial conditions that lead to modulational stability, there is the chance of more predictable dynamics. This is in sharp contrast the focusing NLS which is always unstable. To understand the stability transition for the MNLS equation and its role in obtaining semiclassical asymptotics of the MNLS, we must first understand the forward transform. In recent work with Peter Miller, we have analyzed the forward scattering transform for the MNLS in this regime [15] and proved the following results:

(i) We obtain a condition on initial data for modulational stability.

(ii) We obtain a bound for eigenvalues of the spectral problem corresponding to the forward scattering transform for the MNLS equation.

(iii) We calculate the scattering data explicitly for a certain multi-parameter family of initial conditions.

To obtain these results, techniques from different areas of mathematical analysis were used including:

- Methods for PDE and Ordinary Differential Equations (ODE)
- Spectral Theory
- Singular and non-singular integral equations
- Complex Analysis
- Special Functions

The stability condition was obtained by calculating the so-called “modulation equations” for the MNLS. This involves considering solutions to the MNLS of the form \(\phi(x,t) = A(x,t)e^{iS(x,t)/\varepsilon}\), where \(A(x,t)\) and \(S(x,t)\) are real valued functions. We then substitute this into the MNLS, arriving at a system for \(A\) and \(S\). Dropping the formally small terms in \(\varepsilon\) gives the modulation equations. One then discovers that if \(\alpha^2A^2 + \alpha S_x - 1 > 0\) when the system has real distinct characteristic velocities and is said to have modulational instability. If \(\alpha^2A^2 + \alpha S_x - 1 < 0\), the system is elliptic and modulationally unstable.

One of the strengths of integrable PDEs is that they can be realized as the as the compatibility condition for a linear system known as the Lax pair. The first equation of this pair has the form \(Lv = \frac{\partial}{\partial x}L\), where \(L\) depends on an auxiliary spectral parameter \(k\) as well as the solution to the MNLS, \(\phi(x,t)\). The forward scattering procedure involves spectral analysis for this equation. For each \(k\) with \(\text{Im}(k^2) \neq 0\), there are two one-dimensional subspaces of solutions that decay to zero as \(x \to \pm \infty\), respectively. Eigenvalues are values of \(k\) for which these two subspaces coincide, and eigenfunctions are the corresponding functions \(v\) for each eigenvalue. This spectral problem is not an eigenvalue problem in the traditional sense as \(k\) appears both linearly and quadratically in \(L\), complicating the spectral theory. Even in the limit when \(\alpha \to 0\), the system approaches the non-self-adjoint system associated to the focusing NLS, which is still quite complicated. Due to the integrability, the eigenvalues do not depend on time and thus one can analyze the spectral problem when \(t = 0\). In the work with Miller, we have obtained a “hyperbolic shadow” bound for the eigenvalues associated to initial conditions of the form \(\phi(x,t = 0) = A(x)e^{iS(x)/\varepsilon}\), where \(A(x)\) and \(S(x)\) are real valued functions that have controlled behavior as \(x \to \pm \infty\).

The “shadow” is cast on a complex turning point curve, which is the collection of complex numbers in the \(k\)-plane for which there exists at least one real turning point from WKB analysis of solutions to the Lax equation. The shadow is perhaps best explained by Figure 1 where, in each of these heuristic plots, “light” is projected on all hyperbolae asymptotic to \(\text{Re}(k) = \pm \text{Im}(k)\) in the direction of the arrows, when light hits the blue turning point curve, the hyperbolae change from black to red and begin to cast the shadow. The entire shadow region is shaded green. The symmetry in the problem gives analogous regions in each quadrant.

While it is difficult to find explicit formulae for eigenvalues associated to general initial conditions there has been some success in computing eigenvalues numerically in the NLS case. In [6], J. Bronski numerically computed the eigenvalues corresponding the the focusing NLS for numerous different initial conditions. In this result, Bronski used a similar “shadow” bound for the focusing NLS equation to determine a region in which to look for the eigenvalues. In future work, I plan to use the bound to obtain numerical results for the location of eigenvalues associated to various initial conditions. Moreover, since the eigenvalues appear as singularities of the associated RHP used in the inverse scattering procedure, it will be important to know their locations as we proceed to analyze the inverse problem.

As mentioned, we calculated explicitly the eigenvalues and all the spectral information for a family of initial conditions of the form mentioned above where \(A(x) = \nu \text{sech}(x)\), and \(S'(x) = \mu \tanh(x) + \delta\), for real constants \(\nu, \mu, \text{and } \delta\), with \(\nu > 0\). The methods used to calculate the spectral information are different from those used for similar results in the focusing NLS case (see [21], [22]). Special function theory plays an important role in calculating the spectral information. We develop integral representations for the solutions to the
Lax equation using hypergeometric functions and then obtaining limiting values for these integrals in terms of Gamma functions. Properties of these special functions allow us to obtain all the spectral information. There has been success in the focusing NLS (see [20] and [22]) in carrying out the inverse theory for cases when the spectral information is known explicitly. Since we now know the spectral data associated to this family of initial conditions for all \( \varepsilon > 0 \), we are equipped to carry out the inverse scattering and consider the semiclassical asymptotics of the solutions to the corresponding Cauchy problem. This family of initial conditions is particularly interesting as certain choices of the constants \( \mu, \nu \), and \( \delta \) and \( \alpha \) give functions that satisfy the stability condition for all \( x \), while for other choices, the condition fails for some or all \( x \). Consequently, further study of the inverse problem for this family can provide even more information about the stability transition. Furthermore, it may be possible to use the results for this family to extrapolate to more general initial conditions that can be compared to this family as was done in [20] for the NLS case.

With the facts we have proven for the forward scattering of the MNLS system, we are now beginning to analyze the inverse problem where we will utilize the tools of inverse scattering and Riemann-Hilbert problems to obtain complete semiclassical asymptotics of solutions to the MNLS equation. The study of this inverse problem is part of ongoing and future research.

**Nonlinear Gibbs-type phenomenon**

In the paper [14], as well as my thesis [13], Kenneth McLaughlin and I examine the defocusing NLS, 
\[ i \partial_t + \partial_x^2 - 2|\phi|^2 \phi = 0, \]
with square-well initial data, \( g(x) = 1 \) for \( x \in (-1, 1) \), and 0 otherwise. For time \( t > 0 \), the solution to this initial value problem becomes smooth. If time is reversed, the solution does not uniformly approach the initial data which has jump discontinuities at \( \pm 1 \). In fact, rapid oscillations develop near the values of \( x = \pm 1 \), for small \( t \). This behavior is analogous to the Gibbs phenomenon for Fourier integrals. Utilizing the Riemann-Hilbert formulation and the steepest decent methods of Deift and Zhou, we describe this behavior in detail by obtaining leading order asymptotics and the first order correction for the solution. While this project began as part of my thesis work, considerable work was done in [14] to streamline the arguments of [13]. In doing so, we developed a second method (still using RHP techniques) to analyze this problem. This second method leads to higher accuracy and will greatly simplify similar analysis for more general initial conditions.

To gain motivation, consider a linear model by dropping the cubic nonlinearity, and using the same initial data. Using Fourier analysis one can write down the solution to this model as a Fourier integral. Namely:

\[
\phi_{\text{lin}}(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(y)e^{-ity^2 + ixy}}{y} dy.
\]

In Figure 2, \( \phi_{\text{lin}} \) is plotted as a function of \( x \) when \( t = .0001 \). The Gibbs phenomenon is evident in the rapid oscillations near \( x = \pm 1 \). Figure 3 is also \( \phi_{\text{lin}} \) when \( t = .0001 \) but here it is plotted as a function of \( s = (x-1)(.0001)^{-5} \). The scaling \( s = (x-1)t^{-5} \) becomes vital, for, if this scaling is made, then standard stationary phase/steepest decent arguments for integrals can isolate the rapidly oscillating behavior of \( \phi_{\text{lin}} \) near \( x = \pm 1 \) for small \( t \) in terms of special functions and a correction of order \( t^{3/2} \). In particular for \( x \) near 1 and \( t \) small,

\[
\phi_{\text{lin}}(x,t) = \frac{1}{2} \left( \frac{se^{-ix}}{2} \right)^{1/2} + O(t^{3/2}).
\]
where Erf(x) denotes the standard Error Function. Figure 4 is a plot of the absolute value of the first two terms of right-hand side of the above expansion and clearly captures the oscillations in the full solution in Figure 2.

In order to analyze the nonlinear problem, we used scattering/inverse scattering transform methods. For this particular initial condition g(x), finding the spectral data reduces to solving a constant coefficient system of ordinary differential equations. The time evolution is explicit and easy to obtain from the Lax Pair. As is often the case in such problems, the inverse step can be realized by solving an associated RHP. We construct this RHP then follow and build upon the steepest decent techniques of Deift and Zhou to investigate this RHP when t is small. These methods provide a framework for analyzing a RHP that consists of the following steps:

**Step 1** Perform explicit transformations to equivalent RHPs more suitable in the asymptotic regime of interest.

**Step 2** Construct a *model* RHP that is explicitly solvable in terms of special functions, built to capture the dominant behavior of the deformed original problem.

**Step 3** Carry out detailed error analysis comparing the model and the original problem to arrive at an asymptotic expansion (in this case for small values of t) for the solution to the original RHP.

This is by no means an algorithmic approach, and the details must be carefully tailored to the problem at hand. These techniques incorporate ideas from numerous mathematical areas including:

- Complex analysis and analytic function theory
- Singular integral equations
- Classical analysis
- Special functions
- Potential theory

We then extract (from the asymptotic solution to the RHP) the leading order solution to the posed initial value problem including a first order correction and a uniform estimation of the error. In particular, for the initial condition g(x) (as defined above) the solution φ(x, t) to the defocusing NLS exhibits rapid oscillations for small values of t near x = ±1. The behavior of these oscillations is realized as a nonlinear Gibbs Phenomenon. For K > t > 0, and for all x, the following asymptotic expansion is valid:

\[ φ(x, t) = φ_{\text{lin}}(x, t) + O(t). \]

where \( φ_{\text{lin}}(x, t) \) is the solution to the corresponding linear system with an expansion in powers of \( t^{\frac{1}{2}} \) given above. Additionally, one can write the correction terms explicitly. However, as it is quite long, we omit them here for brevity but they can be found in [14].

As is evident from these results, the *leading order* oscillations for this particular initial condition are essentially described by the linear behavior. However, from preliminary work, it is expected that for a square well initial potential with height given by some large parameter (say \( 1/\varepsilon \) for \( \varepsilon \) small) the nonlinearity will appear earlier. In particular, I expect that time can be scaled with \( \varepsilon \) so that one can recognize the oscillations for these initial conditions in terms of special functions that can be thought of as a nonlinear generalization of the Error function similar to the way in which the Painlevé II function is a nonlinear generalization of the Airy function. I plan to investigate the short time asymptotics for the defocusing NLS (and other systems) for a broader class of initial conditions, including these sequences of square well potentials with large heights. I anticipate that the new methods (particularly those using Hölder bounds) from [14] will be extremely useful in this analysis. Additionally, further analysis of the forward transform must be done to understand the behavior for a *general* class of initial conditions.
Random Matrix Theory and Orthogonal Polynomials

As mentioned earlier, the Riemann-Hilbert techniques used in analyzing integrable PDEs have proved very useful in analyzing problems in random matrix theory and orthogonal polynomials, including several questions of universality for random matrices. The essence of universality for random matrix ensembles is that certain local statistics in limiting regimes are independent of the exact structure of the randomness in the matrix ensemble. Results of this nature include [5], [8], [11], and [19] among others. Additionally, several of the universal distributions appearing in random matrix theory have been proven to describe limit laws for models in combinatorial and probability theory. One of the common universal limiting distributions that appears is known as the Tracy-Widom distribution.

In 1994 Tracy and Widom [23] considered the statistics of the largest eigenvalue of the Gaussian Unitary Ensemble (GUE) of $N \times N$ Hermitian matrices $M = (M_{ij})$ with the probability distribution:

$$
\frac{e^{-\operatorname{Tr} M^2}}{Z_N} \prod_{i=1}^N dM_{ii} \prod_{1 \leq i < j \leq N} d(\operatorname{Re}(M_{ij}))d(\operatorname{Im}(M_{ij}))
$$

where $Z_N$ is a normalization factor. In this work the authors showed that, (after appropriate scaling) the statistics of the largest eigenvalue as $N \to \infty$ were described by what is now referred to as Tracy-Widom distribution $F(x)$. $F(x)$ can be written as a determinant for a certain integral operator built from the so-called Airy kernel (see [23] for details). It can also be written as $\ln(F(x)) = -\int_x^{\infty} (s-x) u(s)^2 ds$, where $u(s)$ solves the Painlevé II equation, and $u(x) \approx -Ai(s)$ as $s \to +\infty$. Using the asymptotics of the Airy function $Ai(s)$, Tracy and Widom calculated the behavior of $F(x)$ as $x \to +\infty$. However, using this formulation of $F$ one would need some global information about $u$ to understand fully the asymptotics as $x \to -\infty$. As shown in [23], one can use the Airy asymptotics to obtain that when $x \to -\infty$,

$$
F(x) = \frac{\tau_o}{(-x)^{1/8}} e^{-\frac{3}{2} \mu |x|^{1/3}} \left( 1 - \frac{3}{26 x^{2}} + \frac{2025}{213 x^{5}} + \ldots \right),
$$

where the constant $\tau_o$ was conjectured but not proven to be $\tau_o = 2 \pi e^{(1)}$. Furthermore, there are similar asymptotic representations and unknown constants for the density functions $F_1(x)$ and $F_2(x)$ which represent the limiting distributions for the values of the largest eigenvalues in the Gaussian Orthogonal Ensemble (GOE) and the Gaussian Symplectic Ensemble (GSE) [24]. In the recent work [9], the authors prove the conjectured value of the constant $\tau_o$ for the GUE case.

In work with Jinho Baik and Robert Buckingham we have obtained an alternate proof of the value of $\tau_o$ for the GUE case and as proved the corresponding results in the GOE and GSE cases as well. The key new idea of the approach utilizes the fact that $F(x)$ appears as the limiting behavior of numerous objects, the set of which we call the “universality class” associated to $F$. Instead of trying to work with $F$ directly, we carefully choose one of the objects from the “universality class” that lends itself to the asymptotic problem at hand. This approach will prove useful in future work for other universal distributions as well.

In [2], Baik, Deift, and Johansson proved that the Tracy-Widom distribution accurately captured the large $N$ behavior of the longest increasing subsequence of a random sequence of the first $N$ integer. In the process they utilized that $F(x)$ can be realized as the limiting behavior of a certain product built up out of norming constants for orthogonal polynomials on the unit circle associated to a varying exponential weight. Moreover, the distributions in the GOE and GSE cases can be related to similarly constructed products as seen in [4]. The orthogonal polynomials can be found via an associated RHP. Our analysis uses the results of [2] and [4] and builds on them by writing the products associated to the desired distributions in a way more suitable for asymptotics as $x \to -\infty$. In order to find these asymptotics, we must calculate higher order corrections to the estimates obtained for orthogonal polynomials in [2] as well as verify the asymptotics for different regimes of the polynomials as well. To do this, we analyze the RHP associated to these orthogonal polynomials. These asymptotics of the RHP are then used to calculate the limiting behavior of the products and consequently the distributions of interest. The Riemann-Hilbert techniques used encompass techniques from the areas of mathematics mentioned before, along with techniques involving classical orthogonal polynomials, Toeplitz determinants, and numerous ideas from random matrix theory.

Future Directions

As indicated, there are several related questions to the problems mentioned above as well as some others that I plan to investigate, including:

- In ongoing work, I am considering nonlinear Gibbs type phenomenon for other integrable PDE’s as well as more general classes of initial conditions
- I plan to further analyze the forward scattering transform associated to the MNLS, including calculating spectral information explicitly and/or numerically for other initial conditions.
• In work with Peter Miller we are carrying out the inverse spectral analysis for the MNLS with the goal of obtaining complete semiclassical asymptotics for solutions to the MNLS equation. We plan to investigate the impact of the stability condition in the inverse transform and anticipate it will play a significant role in analyzing the associated RHP.

• I plan to calculate the asymptotic behavior for other universal distributions including the universal distribution describing the limiting statistics for the hard edge of the spectrum in various matrix ensembles which can be built out of Bessel functions. A similar constant problem I plan to investigate is related to the asymptotic behavior of a system in statistical mechanics, known as the six-vertex model.

• In work with Guadalupe Teran-Lozano, we plan to exploit this connection and use Riemann-Hilbert techniques to find strong asymptotics for the Ablowitz-Ladik (AL) chain in the semiclassical limit. The Ablowitz-Ladik chain can be thought of as a discrete version of the focusing NLS. The AL chain is intimately related to discrete orthogonal polynomials on the unit circle.

• I plan to consider other asymptotic regimes for the NLS hierarchy as well as other integrable nonlinear PDEs. In addition to Gibbs type phenomenon and semiclassical asymptotics, other asymptotic regimes to consider include rigorously weakly nonlinear asymptotics and long time asymptotics.

• I plan to expand my knowledge of random matrices including the study the connections between universal distributions and limit laws for models from combinatorics, probability theory, and statistical mechanics.

References


