Review Outline

Definitions

- $L_2$, $\|\cdot\|$, $\langle \cdot , \cdot \rangle$, orthogonal, orthonormal, basis, Sturm-Liouville Problem, eigenvalue, eigenfunction, steady state solution, generalized Fourier Series, generalized Fourier Coefficients.

Theorems/Concepts to understand. Proofs that you need to know are stated.

- For any hermitian symmetric inner product know the following:
  
  (a) Be able to verify that $\|f + g\|^2 = \|f\|^2 + 2Re \langle f, g \rangle \|g\|^2$.
  
  (b) Know Cauchy’s inequality
  
  (c) Know the Triangle inequality and how to prove it using (a) and (b).
  
  (d) Know Bessel’s Inequality
  
  (e) Know the Pathagorean Theorem and how to prove it.

- Know what the statement “$L^2(a, b)$ is a complete set” means. Know an example of sequence in $PS(a, b)$ that converges to something that is not in $PS(a, b)$. (In other words show that $PS(a, b)$ is not a complete space).

- For an orthonormal/orthogonal set $\{\phi_n(x)\}$ know the three equivalent conditions any of which imply that $\{\phi_n(x)\}$ is an orthonormal/orthogonal basis of $L^2(a, b)$.

- Know how to prove that the three conditions mentioned in the previous bullet are indeed equivalent.

- Assuming the facts that we know about uniform convergence of Fourier Series in $PS(a, b)$ be able to sketch the proof that $\{e^{inn\theta}\}_{n=-\infty}^{\infty}$ is a basis in $L^2(-\pi, \pi)$.

- Know what the Dominated Convergence Theorem says.

- Given a sequence of functions $f_n(x)$ know what it means to say that $f_n$ converges to $f(x)$ pointwise, uniformly, and in norm. Know examples of:

  (a) $f_n(x) \to f(x)$ pointwise but not uniformly. Be able to show why the Dominated Convergence Theorem fails for this case.
  
  (b) $f_n(x) \to f(x)$ in norm but not pointwise.

- Know what a Regular Sturm-Liouville Problem looks like.

Skills:

- Be able to compute $\|\cdot\|$ and $\langle \cdot , \cdot \rangle$ mean for objects in $\mathbb{R}^k$, $\mathbb{C}^k$ and $L^2(a, b)$.

- Be able to verify if set of functions is orthonormal/orthogonal in $L^2(a, b)$.

- Know how to expand functions as a generalized Fourier Series.

- Know how to find the eigenvalues and eigenvectors of a Regular Sturm Liouville Problem for $a \leq x \leq b$. Be able to state that the eigenfunctions of such a problem give you an orthogonal basis on $L^2(a, b)$.

- Know how to show a set $\phi_n$ is a basis of $L^2(a_1, b_1)$ based on the fact that some other related set $\psi_n$ is a basis on some related space $L^2(a_2, b_2)$ Examples of this kind of problem are 4,5,6, and 7 from section 3.3.
• Know Paresval’s Equality and how one might use it to verify the value of certain series (see problem 10 section 3.3).

• Given an orthonormal/orthogonal set \( \phi_n \) in \( L^2(a,b) \), know how to approximate any function \( f \) as a linear combination of the \( \phi_n \) such that the approximation is the best approximation possible under the \( L^2(a,b) \) norm. When is the approximation actually equal to \( f \).

• Understand the three basic ideas we talked about in solving inhomogeneous boundary value problems.
  
  (a) Split the problem into simpler problems
  
  (b) Expand everything in terms of an orthogonal/orthonormal basis coming from an associated Sturm-Liouville problem.
  
  (c) Find a steady state solution, subtract it off and reduce to a simpler (homogeneous) problem.

**KNOW HOW TO USE THESE TO SOLVE THESE KIDS OF PROBLEMS** (problems like those from 4.2, 4.3 and 4.4 and those that we did outside on 5/16.

• Be able to identify the physical interpretation of the boundary and initial conditions for the problems mentioned above.

• Be able to identify the long time asymptotic behavior of the solutions to the problems mentioned above.