Some Review Problems
Here are a few problems that are practice using the techniques we have developed in class. These are by no means all of the types of problems that will appear on the exam but they are good practice for the kinds of problems from chapter 4 and one (the last one) from chapter 5. After the problems are some more things you should know from Chapter 5 along with everything else that was outlined from the previous two exams. The last page of this file contains a bunch of information that will be given to you on the final.

1. Solve for $u(x, t)$.

$$u_t = 4u_{xx}$$
$$u(0, t) = 5$$
$$u(\pi, t) = 0$$
$$u(x, 0) = 5$$

2. Solve for $u(x, t)$.

$$u_t = u_{xx} + u$$
$$u(0, t) = 0$$
$$u(\pi, t) = 0$$
$$u(x, 0) = \sin(x)$$

3. Solve for $u(x, y)$.

$$u_{xx} + u_{yy} = 0$$
$$u(0, \pi) = 0$$
$$u(x, -\pi) = 0$$
$$u(\pi, y) = \sin(y)$$
$$u(-\pi, y) = 0$$

4. Solve for $u(x, t)$.

$$u_t = u_{xx} + t$$
$$u_x(0, t) = 0$$
$$u_x(L, t) = 0$$
$$u(x, 0) = 0$$

5. Solve for $u(x, t)$.

$$u_t = u_{xx} + x$$
$$u(0, t) = 1$$
$$u(L, t) = 0$$
$$u(x, 0) = 0$$
6. Solve for \( u(r, \theta, z) \) on \( D = \{(r, \theta, z)|0 \leq r \leq 1, 0 \leq z \leq 1\} \).

\[
0 = u_{rr} + \frac{1}{r} u_r + u_{zz}
\]
\[
u(1, \theta, z) = 0
\]
\[
u(r, \theta, 0) = 0
\]
\[
u(r, \theta, 1) = r^2
\]

7. Solve for \( u(r, \theta, t) \) on \( D = \{(r, \theta)|0 \leq r \leq 1, -\pi \leq \theta \leq \pi\} \).

\[
u_t = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta}
\]
\[
u(1, \theta, t) = 0
\]
\[
u(r, \theta, 0) = (r - 1) \cos(2\theta)
\]

Things to know from Chapter 5.

- Know how to derive the Laplacian in polar coordinates.
- Be able to identify Bessel’s equation.
- Know how to use the recurrence relations for \( J_\nu(x) \) to derive values/integrals/etc.
- Know what \( L_2^2(0, b) \) means and how to expand a function in \( L_2^2(0, b) \) in terms of Bessel functions and the recurrence relations (like the problems from 5.4).
- Be able to solve PDE’s in polar coordinates using Bessel functions (like the last problem in the list above, the problems done in class during the last week and the problems from section 5.5).
- Know approximately the behavior of \( J_\nu(x) \) at \( x = 0 \) and \( x \to \infty \).
Recurrence Formulas. For all $x$ and $\nu$

$$\frac{d}{dx} [x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x),$$

$$\frac{d}{dx} [x^{\nu} J_\nu(x)] = x^{\nu} J_{\nu-1}(x),$$

$$x J'_{\nu}(x) - \nu J_\nu(x) = -x J_{\nu+1}(x)$$

$$x J'_{\nu}(x) + \nu J_\nu(x) = x J_{\nu-1}(x)$$

$$x J_{\nu-1}(x) + x J_{\nu+1}(x) = 2\nu J_\nu(x),$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2 J'_{\nu}(x).$$

**Theorem 5.3.** Suppose $\nu \geq 0$, $b > 0$ and $w(x) = x$.

(a) Let $\{\lambda_k\}_{1}^{\infty}$ be the positive zeros of $J_\nu(x)$, and let $\phi_k(x) = J_\nu(\lambda_k x/b)$. Then $\{\phi_k\}_{1}^{\infty}$ is an orthogonal basis for $L_w^2(0, b)$, and

$$||\phi_k||^2_w = \frac{b^2}{2} J_{\nu+1}(\lambda_k)^2$$

(b) Suppose that $c \geq -\nu$. Let $\{\tilde{\lambda}_k\}_{1}^{\infty}$ be the positive zeros of $cJ_\nu(x) + xJ'_\nu(x)$, and let $\psi_k(x) = J_\nu(\tilde{\lambda}_k x/b)$.

(i) If $c > -\nu$ then $\{\psi_k\}_{1}^{\infty}$ is an orthogonal basis for $L_w^2(0, b)$.

(ii) If $c = -\nu$ then $\{\psi_k\}_{0}^{\infty}$ is an orthogonal basis for $L_w^2(0, b)$, where $\psi_0(x) = x^\nu$.

Moreover,

$$||\psi_k||^2_w = \frac{b^2(\tilde{\lambda}_k^2 - \nu^2 + c^2)}{2\lambda_k^2} J_{\nu}(\tilde{\lambda}_k)^2, \quad k \geq 1,$$

$$||\psi_0||^2_w = \frac{b^{2\nu+2}}{2\nu + 2}.$$

**Theorem 5.3.** Let $\lambda_{1,n}, \lambda_{2,n}, \lambda_{3,n}, \ldots$ be the positive zeros of $J_n(x)$. Then

$$\left\{ J_n \left( \frac{\lambda_{n,n}}{b} \right) \cos(n\theta) : n \geq 0, k \geq 1 \right\} \cup \left\{ J_n \left( \frac{\lambda_{n,k}}{b} \right) \sin(n\theta) : n, k \geq 1 \right\}$$

is an orthogonal basis for $L^2(D)$ where $D$ is the disc of radius $b$ about the origin. Moreover if

$$f(r, \theta) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} (c_{nk} \cos(n\theta) + d_{nk} \sin(n\theta)) J_n \left( \frac{\lambda_{kn}}{b} \right),$$

then

$$c_{0k} = \frac{1}{\pi b^2 J_0(\lambda_{0,0})^2} \int_{-\pi}^{\pi} \int_{0}^{b} f(r, \theta) J_0 \left( \frac{\lambda_{0,0}}{b} \right) r dr d\theta$$

$$c_{nk} = \frac{2}{\pi b^2 J_{n+1}(\lambda_{n,n})^2} \int_{-\pi}^{\pi} \int_{0}^{b} f(r, \theta) J_0 \left( \frac{\lambda_{n,n}}{b} \right) \cos(n\theta) r dr d\theta$$

$$b_{nk} = \frac{2}{\pi b^2 J_{n+1}(\lambda_{n,n})^2} \int_{-\pi}^{\pi} \int_{0}^{b} f(r, \theta) J_0 \left( \frac{\lambda_{n,n}}{b} \right) \sin(n\theta) r dr d\theta$$