Section 5.5

2. To solve this problem we first need to write down the actual BVP we wish to solve. Here we are dealing with the Heat equation in 3 spacial dimensions (cylindrical coordinates). We are also to assume that the heat constant is equal to 1. Thus we need to consider the equation:

\[ u_t = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz}. \]

The boundary terms tell us that the top and bottom of the cylinder of radius \( \rho + \delta \) and height \( l \) insulated, thus if \( u(r, \theta, z, t) \) is the function we are after then,

\[ u_z(r, \theta, 0, t) = 0, \quad u_z(r, \theta, l, t) = 0. \]

We also have that outside the cylinder is held at constant temperature \( B \) thus

\[ u(\rho + \delta, \theta, z, t) = B. \]

We also know the initial temperature to be:

\[ u(r, \theta, z, 0) = \begin{cases} A & r < \rho \\ B & \rho < r < \rho + \delta \end{cases}. \]

These conditions (along with the implicit assumption that the solution makes sense at \( r = 0 \) and that it is 2\( \pi \)-periodic in \( \theta \) will give the BVP of:

\[ u_t = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} \]

\[ u_z(r, \theta, 0, t) = 0 \]

\[ u_z(r, \theta, l, t) = 0 \]

\[ u(\rho + \delta, \theta, z, t) = B \]

\[ u(r, \theta, z, 0) = \begin{cases} A & r < \rho \\ B & \rho < r < \rho + \delta \end{cases} \]

Now the first thing that I notice is that we have non-zero boundary conditions at \( r = \rho + \delta \). This leads me to look perhaps for a steady state solution. Now that could be a function of all \( r, \theta \) and \( z \) (but not \( t \)). However, based on the problem at hand I see that the boundary values do not depend on \( z \) or \( \theta \), so I am going to look for a steady state solution that only depends on \( r \). Such a function would satisfy:

\[ 0 = u''(r) + r^{-1}u'(r), \quad u(\rho + \delta) = B. \]

Clearly any constant function satisfies the equation above and thus the function \( u_0 = B \) is a steady state solution. If I take this solution and subtract it from the solution \( u \) from the original problem then I get that \( v(r, \theta, z, t) = u(r, \theta, z, t) - B \) satisfies the equation:

\[ v_t = v_{rr} + r^{-1}v_r + r^{-2}v_{\theta\theta} + v_{zz} \]

\[ v_z(r, \theta, 0, t) = 0 \]

\[ v_z(r, \theta, l, t) = 0 \]

\[ v(\rho + \delta, \theta, z, t) = 0 \]

\[ v(r, \theta, z, 0) = \begin{cases} A - B & r < \rho \\ 0 & \rho < r < \rho + \delta \end{cases} \]

Now again at this point there are multiple ways to proceed. Here I am going to present the easiest, where I assume that the solution does not depend on either \( z \) or \( \theta \). So we assume that our solution
is of the form $R(r)T(t)$ and substitute this into the equation. Since the solution does not depend on $\theta$ and $z$, then $u_{\theta\theta}$ and $u_{zz}$ are both 0. So we arrive at,

$$T''R = R''T + r^{-1}R'T.$$

Thus,

$$\frac{T'}{T} = \frac{R'' + r^{-1}R'}{R}.$$

Setting both sides equal to $-\mu^2$ gives that:

$$R'' + r^{-1}R' + \mu^2R = 0.$$

If one makes the substitution $f(\mu r) = R(r)$ one sees that $f$ satisfies Bessel’s Equation of order 0. Thus, $R(r) = J_0(\mu r)$. The boundary term tells us that $R(\rho + \delta) = 0$, which forces $J_0(\mu(\rho + \delta)) = 0$ and thus $\mu = \lambda_k/(\rho + \delta)$ where $\lambda_k$ is the $k$th largest positive eigenvalue of the Bessel function of order 0. Form knowledge discussed in class we know that this set of function $J_0(\frac{\lambda_k r}{\rho + \delta})$ is a basis in $L^2(0, \rho + \delta)$. Now looking at the equation for $T$ and using the fact that $-\mu^2 = -\lambda_k^2/(\delta + \rho)^2$ we have that $T(t) = ce^{-t\lambda_k^2/(\delta + \rho)^2}$. Thus we can conclude that our solution for $v$ is of the form:

$$v(r, \theta, z, t) = \sum_{k=1}^{\infty} c_k e^{-\frac{\lambda_k^2}{\rho + \delta} \frac{\rho}{\rho + \delta}} J_0\left(\frac{\lambda_k r}{\rho + \delta}\right).$$

Using the initial condition we must have that,

$$\begin{cases}
A - B & r < \rho \\
0 & \rho < r < \rho + \delta
\end{cases} \sum_{k=1}^{\infty} c_k e J_0\left(\frac{\lambda_k r}{\rho + \delta}\right).$$

Thus:

$$c_k = \frac{\int_0^\rho (A - B)r J_0\left(\frac{\lambda_k r}{\rho + \delta}\right) dr}{||J_0\left(\frac{\lambda_k r}{\rho + \delta}\right)||^2}.$$

From the theorem proved in class,

$$||J_0\left(\frac{\lambda_k r}{\rho + \delta}\right)||^2 = \frac{(\rho + \delta)^2}{2} J_1(\lambda_k)^2.$$

Also,

$$(A - B) \int_0^\rho r J_0\left(\frac{\lambda_k r}{\rho + \delta}\right) dr = \frac{(A - B)(\rho + \delta)^2}{\lambda_k^2} \int_0^{\rho\lambda_k} s J_0(s) ds = \frac{(A - B)(\rho + \delta)^2}{\lambda_k^2} \frac{\lambda_k \rho}{\rho + \delta} J_1\left(\frac{\lambda_k \rho}{\rho + \delta}\right).$$

In the last equality I have used the fact that $s J_0(s) = [s J_1(s)]'$. Consequently we have that:

$$c_k = \frac{\frac{(A - B)\rho(\rho + \delta)}{\lambda_k}}{\frac{(\rho + \delta)^2}{2} J_1(\lambda_k)^2} J_1\left(\frac{\lambda_k \rho}{\rho + \delta}\right) = \frac{2(A - B)\rho J_1\left(\frac{\lambda_k \rho}{\rho + \delta}\right)}{\lambda_k(\rho + \delta) J_1(\lambda_k)^2}.$$

Thus our series solution for $v$ is given by:

$$v(r, \theta, z, t) = \sum_{k=1}^{\infty} \frac{2(A - B)\rho J_1\left(\frac{\lambda_k \rho}{\rho + \delta}\right)}{\lambda_k(\rho + \delta) J_1(\lambda_k)^2} e^{-\frac{\lambda_k^2}{(\rho + \delta)^2}} J_0\left(\frac{\lambda_k r}{\rho + \delta}\right).$$

Finally we go back to $u(r, \theta, z, t)$ and we have,

$$u(r, \theta, z, t) = B + \sum_{k=1}^{\infty} \frac{2(A - B)\rho J_1\left(\frac{\lambda_k \rho}{\rho + \delta}\right)}{\lambda_k(\rho + \delta) J_1(\lambda_k)^2} e^{-\frac{\lambda_k^2}{(\rho + \delta)^2}} J_0\left(\frac{\lambda_k r}{\rho + \delta}\right),$$

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