Section 2.1

4. Verify the formulas for entries 4 and 16 in Table 1 in Section 2.1 by hand using the definition of the Fourier coefficients introduced in this section.

Recall that, for a $2\pi$-periodic, integrable function $f$ we have

\[
 a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta,
\]

\[
 b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta.
\]

Entry 4 claims that the Fourier series for $f(\theta) = \begin{cases} 0 & \text{if } -\pi < \theta < 0 \\ \theta & \text{if } 0 \leq \theta < \pi \end{cases}$ is given by

\[
 f(\theta) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta.
\]

In terms of Fourier coefficients, the claim is that, for this particular $f$, we have

\[
 a_0 = \frac{\pi}{2}, \quad a_n = \begin{cases} -\frac{2}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}, \quad b_n = \frac{(-1)^{n+1}}{n}.
\]

We now verify these values using the definitions for the $a_n$ and $b_n$ given above.

\[
 a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos 0 \, d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta = \frac{1}{\pi} \int_{0}^{\pi} \theta \, d\theta = \frac{1}{\pi} \left( \frac{\theta^2}{2} \bigg|_{\theta=0}^{\pi} \right) = \frac{\pi}{2},
\]

as we’d like. For $n > 0$, we have the following:

\[
 a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos n\theta \, d\theta = \frac{1}{\pi} \left( \frac{\theta \sin n\theta}{n} \bigg|_{\theta=0}^{\pi} - \int_{0}^{\pi} \frac{1}{n} \sin n\theta \, d\theta \right)
\]

\[
 = \begin{cases} 0 & \text{if } n \text{ is even}, \\ -\frac{2}{\pi n} & \text{if } n \text{ is odd}. \end{cases}
\]

Finally, for the $b_n$ we have

\[
 b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta = \frac{1}{\pi} \int_{0}^{\pi} \theta \sin n\theta \, d\theta
\]

\[
 = \frac{1}{\pi} \left( -\frac{\theta}{n} \cos n\theta \bigg|_{\theta=0}^{\pi} + \int_{0}^{\pi} \frac{1}{n} \cos n\theta \, d\theta \right)
\]

\[
 = \frac{1}{\pi} \left( -\frac{\theta}{n} \cos n\theta \bigg|_{\theta=0}^{\pi} + \int_{0}^{\pi} \frac{1}{n} \sin n\theta \, d\theta \right)
\]

\[
 = -\frac{1}{\pi n} \cos n\pi = \frac{(-1)^{n+1}}{n},
\]

as desired.

16. Entry 16 claims that the Fourier series for the function $f(\theta) = \theta^2$ is given by

\[
 f(\theta) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta.
\]
In terms of Fourier coefficients, the claim is that

\[ a_0 = \frac{2\pi^2}{3}, \quad a_n = 4\frac{(-1)^n}{n^2}, \quad b_n = 0. \]

In the case of \(a_0\), we compute

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos 0 \, d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \, d\theta
= \frac{1}{\pi} \left[ \frac{\theta^3}{3} \right]_{\theta=-\pi}^{\pi} = \frac{2\pi^2}{3},
\]

as desired. For \(n > 0\), we find

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \cos n\theta \, d\theta
= \frac{1}{\pi} \left( \frac{2\theta}{n^2} \cos n\theta \right)_{\theta=-\pi}^{\pi} = \frac{2}{n^2} \cos n\pi - \frac{2}{n^2} \cos(-n\pi) = 4\frac{(-1)^n}{n^2},
\]

as desired.

For the \(b_n\), we have

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \sin n\theta \, d\theta
= \frac{1}{\pi} \left( \frac{-\theta^2}{n} \cos n\theta + \frac{2}{n^3} \sin n\theta \right)_{\theta=-\pi}^{\pi}
= 0,
\]

since \(\cos n\pi = \cos(-n\pi)\). Hence we have our coefficients, as desired.

**Section 2.2**

1. Which of the following functions are continuous, piecewise continuous, or piecewise smooth on \([-\pi, \pi]\)? First I will note that any function that is continuous is certainly piecewise continuous.

   (a) \(f(\theta) = \csc \theta\). This function has an asymptote at \(\theta = 0\) and thus is not Continuous, PC or PS.

   (b) \(f(\theta) = (\sin \theta)^{1/3}\). This function is Continuous, but not PS since the derivative \((1/3)(\sin \theta)^{-2/3} \cos(\theta)\) has an asymptote at \(\theta = 0\).

   (c) \(f(\theta) = (\sin \theta)^{4/3}\).

   This function is continuous, and the derivative \((4/3)(\sin \theta)^{1/3} \cos(\theta)\) is also continuous.

   (d) \(f(\theta) = \cos \theta\) if \(\theta > 0\), \(f(\theta) = -\cos \theta\) if \(\theta \leq 0\).

   This function is not continuous as the limit from the left as \(\theta\) approaches 0 is not the same as the limit from the left. It is piecewise continuous though as both the left hand and the right hand limits do exist at those values. Also the derivative is not defined at \(\theta = 0\) either, but it does have limits as it approaches those values (in fact both the right and left hand limit are 0).
(e) \( f(\theta) = \sin \theta \) if \( \theta > 0 \), \( f(\theta) = \sin 2\theta \) if \( \theta \leq 0 \).

This function is continuous as both the left hand and the right hand limits at \( \theta = 0 \) are 0. The derivative however is not continuous as it is \( \cos \theta \) for \( \theta > 0 \) and \( 2 \cos \theta \) for \( \theta < 0 \). Consequently the limit from the left of the derivative is 1 while the limit from the right is 2, but these limits do exist and are not infinite so the function is Piecewise smooth.

(f) \( f(\theta) = (\sin \theta)^{1/5} \) if \( \theta < \pi/2 \), \( f(\theta) = \cos \theta \) if \( \theta \geq \pi/2 \). This function is not continuous since \((\sin(\pi/2)^{1/5}) = 1 \neq 0 = \cos(\pi/2)\). However the right and left hand limits do exist at \( \theta = \pi/2 \) and thus the function is Piecewise Continuous. It is not piecewise smooth though. The derivative to the left of \( \pi/2 \) is given by \((1/5)(\sin\theta)^{-4/5}\cos\theta\) has an asymptote at \( \theta = 0 \).

2. For all of these problems we use the fact that the Fourier Series converges to the value:

\[ \frac{1}{2} \left( f(\theta^+) + f(\theta^-) \right). \]

This can be interpreted as the average value of the limits from either side of the function \( f(\theta) \) at the jump at any discontinuities.

(a) For entry 6 on the table there are jumps at any multiple of \( \pi \). At even multiples of \( \pi \) the limit of the function from the right is 1 and the limit from the left is -1. Thus the Fourier series will converge to the average of these values which is 0. At off multiples of \( \pi \) the situation is reversed but the average is still 0.

(b) For entry 7 on the table there are discontinuities at every multiple of \( \pi \). The limit from the right is 0 (or 1) and the limit from the left is 1 (or 0) consequently the average value of these limits is 1/2 at any multiple of \( \pi \) and consequently this is what the function converges to at these points.

(c) For entry 12 on the table the jumps occur at \(-a + 2k\pi \) or \(a + 2k\pi \) for any integer \( k \). At any of these points the jump goes from 0 to \((2a)^{-1}\) or from \((2a)^{-1}\) to 0. In either case the average value is \((4a)^{-1}\). This is the value that the Fourier Series converges to at the points of discontinuity.

(d) For entry 18 on the table there are jump discontinuities at odd multiples of \( \pi \) (or \((2k + 1)\pi \) for integers \( k \)). At each of these points the limit from the left is \( e^{b\pi} \) and the limit from the left is \( e^{-b\pi} \). Consequently the average of these is \((e^{b\pi} + e^{-b\pi})/2 \) or \( \text{sech}(b\pi) \). This number is the value to which the Fourier Series will converge at the odd multiples of \( \pi \).

3. This problem asks you to show that:

\[ \sum_{1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}, \]

and

\[ \sum_{1}^{\infty} \frac{(-1)^{n+1}}{4n^2 + 1} = \frac{\pi - 2}{4}. \]

To prove these formula we could use multiple different formulae from the table, but I will choose to work with entry 8 that says:

\[ |\sin(\theta)| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{1}^{\infty} \frac{\cos(2n\theta)}{4n^2 - 1}. \]

If I evaluate this formula at 0 then I obtain:

\[ 0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{1}^{\infty} \frac{1}{4n^2 - 1}. \]
By subtracting the series to both sides, and then multiplying by $\pi/4$ we obtain the first formula we wanted. Alternatively, if I evaluate the formula at $\theta = \pi/2$ and use the fact that $\cos(2n\pi/2) = \cos(n\pi) = (-1)^n$ I obtain:

$$1 = |\sin(\pi/2)| = \frac{2}{\pi} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

Multiplying both sides by $\pi/4$ gives:

$$\frac{\pi}{4} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$ 

Algebra gives:

$$\frac{\pi}{4} - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1}.$$ 

If I combine the terms on the left hand side I will get the second formula that I desired.

5. This problem asks us to show that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$$

To show this we use formula 17 from the table which says that for $-\pi < \theta < \pi$,

$$\theta(\pi - \theta) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{(2n-1)^3}$$

I want $\sin((2n-1)\theta)$ to be equal to $(-1)^{n+1}$. I know that $\sin(\pi/2) = 1$, $\sin(3\pi/2) = -1$ etc. So this says that $\sin((2n-1)n\pi/2) = (-1)^{n+1}$. So we evaluate the formula above at $\pi/2$. This gives:

$$\frac{\pi^2}{4} = \frac{\pi}{2} \frac{\pi}{2} = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

Multiplying both sides by $\pi/8$ gives the desired formula.

Section 2.3

1. Derive the result of entry 16 of Table 1 by using equation (2.17) and Theorem 2.4.

Entry 16 states that

$$f(\theta) = \theta^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta).$$

Equation (2.17) states that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta) = \frac{\theta}{2}$$

which we can also write as

$$4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta) = 2\theta$$

Theorem 2.4 is an integration result. The function $g(\theta)$ is piecewise continuous (as a periodically defined function). The Fourier coefficients for $g(\theta)$ coming from the above formula are $a_n = 0$ and
\[ b_n^g = 4(-1)^{n+1}/n. \] (It is important that \( a_0 = 0 \) why?) The idea here is to recognize that \( f(\theta) \) is the antiderivative of \( g(\theta) = 2\theta \). Namely,

\[ f(\theta) = \int_0^\theta g(\phi) \, d\phi. \]

(Recall the Fundamental Theorem of calculus). Theorem 2.4 tells us how to obtain Fourier coefficients for \( f(\theta) \) in terms of the coefficients of \( g(\theta) \). Namely, Theorem 2.4 tells us that the Fourier coefficients for \( f(\theta) \) \((\{a_n^f\} \text{ and } \{b_n^f\})\) are given by

\[
\begin{align*}
  a_0^f &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta, \\
  a_n^f &= \frac{-b_n^g}{n}, \quad \text{and} \\
  b_n^f &= \frac{a_n^g}{n}.
\end{align*}
\]

If I plug in the values for Fourier coefficients for \( g \) I get:

\[
\begin{align*}
  a_0^f &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \, d\theta = \frac{2\pi^2}{3}, \\
  a_n^f &= \frac{-4(-1)^{n+1}}{n^2}, \\
  b_n^f &= 0.
\end{align*}
\]

Putting this together we obtain:

\[
\begin{align*}
  f(\theta) &= \frac{a_0^f}{2} + \sum_{n=1}^{\infty} \left( a_n^f \cos(n\theta) + b_n^f \sin(n\theta) \right) \\
  &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta),
\end{align*}
\]

as desired.

2. (c) Starting from entry 16 of Table 1 and using Theorem 2.4, show that \( \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \)

Entry 16 states that

\[ f(\theta) = \theta^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta. \]

Theorem 2.4 is an integration result that, among other things, places more factors of \( n \) in the denominators of our Fourier coefficients. By applying it twice to \( f(\theta) \), we hope to get an expression that can be manipulated into the desired result.

Begin with the formula \( \theta^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta \), subtracting the constant from both sides we have:

\[ \theta^2 - \frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta. \]

The reason why we do this is that Theorem 2.4 is most easily applied when there is no constant term. Applying Theorem 2.4, gives

\[
\begin{align*}
  \theta^3 - \frac{\pi^2}{3} \theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi^3 \, d\phi - \frac{\pi^2}{3} \phi \bigg|_{\phi=-\pi}^{\phi=\pi} + \sum_{n=1}^{\infty} 4 \frac{(-1)^n}{n^3} \sin n\theta \\
  &= \frac{1}{2\pi} \left( \frac{\phi^4}{12} \bigg|_{\phi=-\pi}^{\phi=\pi} \right) + \sum_{n=1}^{\infty} 4 \frac{(-1)^n}{n^3} \sin n\theta \\
  &= \sum_{n=1}^{\infty} 4 \frac{(-1)^n}{n^3} \sin n\theta.
\end{align*}
\]
This by the way proves part (a) of this problem. Note that the result has no constant term in the series on the right, so I can apply the theorem again without any modification giving:

\[
\frac{\theta^4}{12} - \frac{\pi^2}{6} \theta^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi^4 \frac{\phi^2}{12} d\phi - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^4} \cos n\theta
\]

\[
= \frac{1}{2\pi} \left( \frac{\phi^5}{60} - \frac{\pi^2}{18} \phi^3 \bigg|_{-\pi}^{\pi} \right) - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^4} \cos n\theta
\]

\[
= \frac{\pi^4}{60} - \frac{\pi^4}{18} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^4} \cos n\theta.
\]

This proves part (b) of this problem. Now, if we take \(\theta = \pi\), a convenient thing happens: \(\cos n\pi\) is equal \((-1)^n\), and since \((-1)^n(-1)^n = (-1)^{2n} = 1\), then we arrive at the equality

\[
\frac{\pi^4}{12} - \frac{\pi^2}{6} \theta^2 = \frac{\pi^4}{60} - \frac{\pi^4}{18} - 4 \sum_{n=1}^{\infty} \frac{1}{n^4}.
\]

Solving for \(\sum_{n=1}^{\infty} \frac{1}{n^4}\) gives the desired result:

\[
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{4} \left( \frac{1}{60} - \frac{1}{18} - \frac{1}{12} + \frac{1}{6} \right) = \frac{\pi^4}{4} \frac{2}{45} = \frac{\pi^4}{90},
\]

as desired.

4. By entry 8 of Table 1, we have

\[
\sin \theta = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \cos(2n\theta) \frac{4n^2}{4n^2 - 1} \quad (0 \leq \theta \leq \pi)
\]

and we also have

\[
\cos(\theta) = \frac{d}{d\theta} \sin \theta = -\int_{\frac{\theta}{2}}^{\theta} \sin \phi d\phi
\]

If I take the derivative of the Fourier Series for \(\sin\) that is given above then Theorem 2.3 will lead me to:

\[
\cos \theta = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{-2n \sin(2n\theta)}{4n^2 - 1} \quad (0 \leq \theta \leq \pi)
\]

If I take the series given for \(\sin\) above and subtract \(2/\pi\) and then integrating term by term will tell me that:

\[
-\int_{\pi/2}^{\theta} \sin \phi - \frac{2}{\pi} \phi d\phi = \frac{1}{2} A_o + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2 - 1)} \quad (0 \leq \theta \leq \pi)
\]

\[
\cos(\theta) + \frac{2}{\pi} \theta - 1 = \frac{1}{2} A_o + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2 - 1)} \quad (0 \leq \theta \leq \pi)
\]

The constant \(A_o\) can be computed by:

\[
\frac{1}{2} A_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\theta) + \frac{2}{\pi} \theta d\theta = 0
\]
So we have that on one hand by integrating we now have:

\[
\cos(\theta) = 1 - \frac{2}{\pi} \theta + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2 - 1)} \quad (0 \leq \theta \leq \pi)
\]

but by differentiating we have that:

\[
\cos \theta = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{-2n \sin(2n\theta)}{4n^2 - 1} \quad (0 \leq \theta \leq \pi)
\]

So how are these the same thing? Well consider the formula given on line (2.17). This formula states:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta) = \frac{\theta}{2} \text{ for } -\pi \leq x \leq \pi
\]

Also if we take 1 on \((0, \pi)\) and extend it as an odd function and write the corresponding sin series we will have that for \((0, \pi)\) we have from table 1 that:

\[
1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1}
\]

Consequently for \(\theta \in (0, \pi)\) we have that:

\[
\cos \theta = 1 - \frac{2}{\pi} \theta + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2 - 1)} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\theta) + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2 - 1)}
\]

Now I take the middle term on the right hand side and write it as two terms, splitting up the even and odd terms.

This gives me that:

\[
\cos \theta = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n} \sin(2n\theta) + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{-1}{2n-1} \sin((2n-1)\theta) + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2 - 1)}
\]

The first and the third terms of this sum cancel! and the other two combine to give:

\[
\cos \theta = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)} \left( \frac{1}{2n} \sin(2n\theta) \right) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{4n^2}{(4n^2 - 1)} \frac{\sin(2n\theta)}{2n} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{2n \sin(2n\theta)}{(4n^2 - 1)}
\]

Which is exactly the same formula you get if you differentiate.

6. The entries 11 and 12 in table 1 say that for \(-\pi < \theta < \pi\):

\[
f(\theta) = \begin{cases} 
\theta & (-a < \theta < a) \\
\frac{\pi - \theta}{\pi + a} & (a < \theta < \pi) \\
\frac{a + \theta}{\pi + a} & (-\pi < \theta < -a)
\end{cases} \quad \frac{2}{\pi - a} \sum_{n=1}^{\infty} \frac{\sin(na)}{n^2} \sin(n\theta)
\]

\[
g(\theta) = \begin{cases} 
(2a)^{-1} & (|\theta| < a) \\
0 & (a < |\theta| < \pi)
\end{cases} \quad \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(na)}{na} \cos(n\theta)
\]

If one evaluates the derivative of \(f\) one would obtain:
To be sure that this is correct I have used the fact that $f$ is clearly continuous and piecewise smooth, so the derivative holds for all points where the derivative is continuous. So the above formula does not hold at the points of discontinuity of $f$, but that is ok since $f'$ is piecewise continuous. Now if I multiply all of the above equation by $(\pi - a)/(2a\pi)$ I obtain:

$$f'(|\theta|) = \begin{cases} \frac{\pi - a}{2a\pi} & (-a < \theta < a) \\ \frac{1}{2\pi} & (a < |\theta| < \pi) \end{cases}$$

Now adding $1/(2\pi)$ to both sides will give me that:

$$f'(|\theta|) = \begin{cases} \frac{1}{2\pi} & (-a < \theta < a) \\ 0 & (a < |\theta| < \pi) \end{cases}$$

Consequently we have shown that (a) $\frac{1}{2\pi} + \frac{\pi - a}{2a\pi} f'(|\theta|) = g(\theta)$ and that if we perform the corresponding operations on the series we get the same thing.

7. To investigate the smoothness of the following series we utilize the theorem 2.6 from the text.

(a) If I consider the term

$$\lim_{n \to \infty} \left| n^k c_n \right| = \lim_{n \to \infty} \left| \frac{n^k e^{in\theta}}{n^{13.2} + 2n^3 - 1} \right|$$

This limit will be 0 for any number $k \leq 13.2$. It will equal 1 for $k = 13.2$. If $k = 14$ then the limit is infinite. What this means is that for n large enough $|n^{13.2} c_n| < 2$ for n large enough (say bigger then $N_o$, now for $n \leq N_o$ there are a finite number of terms and that can be bounded by a constant. So, there exists a constant M such that $|n^{13.2} c_n| < M$ for all $n$. Or $|c_n| < M n^{-(12 + 1.2)}$. Applying Theorem 2.6 we have that the corresponding $f$ is of class $C^{(12)}$, or that it has all of its first 12 derivatives and they are all continuous.

(b) For the series $f(\theta) = \sum_{n=0}^{\infty} \frac{\cos(n\theta)}{n^{2n}}$. We look at $a_n = 2^{-n}$. The series $|n^k 2^{-n}| \to 0$ for ANY positive value of $k$. One can show this using L'Hospitals rule many times.

$$\lim_{n \to \infty} \frac{n^k}{2^n} = \lim_{n \to \infty} \frac{kn^{k-1}}{ln(2)2^n} = \lim_{n \to \infty} \frac{k!}{(\ln(2))^n n^2} = 0$$

Consequently one could bound $|b_n| < C n^k$ for any positive integer $k$ and since the $a_n = 0$ then we can conclude that $f(\theta)$ is infinitely differentiable.

(c) To investigate the function

$$f(\theta) = \sum_{n} \frac{\cos(2^n \theta)}{2^n}$$

we have to recognize that this is really a series where $b_j = \frac{1}{2}$ whenever $j = 2^n$, and that $b_j = 0$ whenever $j$ is not a multiple of 2. All of the $a_j$ are zero since there are no sine terms. For the nonzero $b_j$ we have that:

$$\lim_{j \to \infty} |j^k b_j| = \lim_{j \to \infty} |j^{k-1}| = \begin{cases} 0 & k < 1 \\ 1 & k = 1 \\ \infty & k > 1 \end{cases}$$
This means that the largest $k$ for which $|b_j| < C_j^{-k}$ is $k \leq 1$, or $|b_j| < C_j^{-(1+0)}$. Since the $a_n$ are all zero, then we can conclude that $f(\theta)$ is in $C^{(0)}$ which is the space of continuous functions. Now from the theorem we can conclude that it is NOT $C^{(1)}$, but we can not conclude that it is $C^{(0)}$ quite either, the best we could do would be to say that $f$ is Piecewise continuous (the answer in the back of the book is not quite right). You might be able to strengthen this to get that it is $C^{(0)}$ but not from what is in this section.

**Section 2.4**

2. Find the Fourier cosine series and the Fourier sine series of the function $f(\theta) = \pi - \theta$ on the interval $[0, \pi]$. To what values do these series converge when $\theta = 0$ and $\theta = \pi$?

We first compute the even and odd extensions of $f$:

$$f_{\text{even}}(\theta) = \begin{cases} f(\theta) & 0 \leq \theta \leq \pi \\ f(-\theta) & -\pi \leq \theta < 0 \end{cases}$$

$$= \begin{cases} \pi - \theta & 0 \leq \theta \leq \pi \\ \pi + \theta & -\pi \leq \theta < 0 \end{cases}$$

$$= \pi - |\theta|,$$

$$f_{\text{odd}}(\theta) = \begin{cases} f(\theta) & 0 \leq \theta \leq \pi \\ -f(-\theta) & -\pi \leq \theta < 0 \end{cases}$$

$$= \begin{cases} \pi - \theta & 0 \leq \theta \leq \pi \\ -\pi - \theta & -\pi \leq \theta < 0 \end{cases}.$$  

Since the Fourier cosine series is simply the Fourier series of the even extension of $f$, we can use the table to obtain it without resorting to brutish computations. By Table 1 Entry 2,

$$|\theta| = \frac{\pi}{2} - 4 \sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2}. $$

Since $f_{\text{even}}$ is a function of $|\theta|$, we get that

$$f_{\text{even}}(\theta) = \pi - |\theta|$$

$$= \pi - \left( \frac{\pi}{2} - 4 \sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2} \right)$$

$$= \frac{\pi}{2} + 4 \sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2}. $$

We can perform a similar trick to obtain the Fourier sine series. Namely if we plot $f_{\text{odd}}(\theta)$ one will see that it is exactly the 3rd entry from the Table, thus

$$f_{\text{odd}}(\theta) = \pi g(\theta) - \theta.$$  

Since Table 1 gives us the Fourier series for $g$, as well as for the identity function $\theta \mapsto \theta$, we can produce the Fourier series for $f_{\text{odd}}$ (that is, the Fourier sine series of $f$) as a linear combination of the series of these two functions:

$$f_{\text{odd}}(\theta) = 2 \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}. $$
5. First we construct the Fourier Cosine Series for $\theta^2$, where we are thinking of it on the interval $[0, \pi]$. To do this we construct an even extension of $\theta^2$, this however is then $\theta^2$ on $[-\pi, \pi]$. The Fourier Cosine series is then exactly the same as entry #16 from the table which is:

$$\text{Fourier Cos Series } \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta)$$

To compute the odd extension we need to think of the odd extension of $\theta^2$, but this is not an entry that is easily read off the table. Instead we use the formula that the coefficients for the Fourier Sine Series on $[0, \pi]$ is given by:

$$b_n = \frac{2}{\pi} \int_0^\pi \theta^2 \sin(n\theta) d\theta = \frac{2}{\pi} \left( \frac{\theta^2}{n} \cos(n\theta) + \frac{2\theta}{n^2} \sin(n\theta) + \frac{2}{n^3} \cos(n\theta) \right) \bigg|_0^\pi = \left( \frac{4}{\pi n^3} - \frac{2\pi}{n} \right) (-1)^n - \frac{4}{\pi n^3}$$

Consequently we have that:

$$\text{Fourier Sin Series } \sum_{n=1}^{\infty} \left( \left( \frac{4}{\pi n^3} - \frac{2\pi}{n} \right) (-1)^n - \frac{4}{\pi n^3} \right) \sin(n\theta)$$

7. Compute the Fourier Sin series for $f(x) = 1$ on the interval $[0, 6\pi]$. The formula for the Fourier Sine coefficients for this interval is given by:

$$b_n = \frac{2}{6\pi} \int_0^{6\pi} \sin \left( \frac{n\pi \theta}{6\pi} \right) d\theta = \frac{1}{3\pi} \int_0^{6\pi} \sin \left( \frac{n\theta}{6} \right) d\theta = \frac{2}{\pi n} (1 - (-1)^n) = \left\{ \begin{array}{ll} \frac{1}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{array} \right.$$

Thus the Fourier Sine Series for $f(x) = 1$ is:

$$1 = \sum_{k=1}^{\infty} \frac{4}{\pi (2k-1)} \sin \left( \frac{(2k-1)x}{6} \right)$$

This of course could have been obtained from formula from the table by evaluating it at $\pi/6$.

9. For this function, to compute the fourier cosine coefficients you must use the formula:

$$a_n = \frac{2}{4} \int_0^2 f(x) \cos \left( \frac{\pi xn}{4} \right) dx = \frac{1}{2} \int_0^2 \cos \left( \frac{\pi xn}{4} \right) dx - \frac{1}{2} \int_0^4 \cos \left( \frac{\pi xn}{4} \right) dx$$

$$= \frac{2}{\pi n} \sin \left( \frac{n\pi x}{4} \right) \bigg|_0^2 - \frac{2}{\pi n} \sin \left( \frac{n\pi x}{4} \right) \bigg|_0^4 = \frac{4}{n\pi} \sin \left( \frac{\pi n}{2} \right) = \left\{ \begin{array}{ll} 0 & n \text{ even} \\ \frac{4(-1)^{k+1}}{(2k-1)\pi} & n = 2k - 1 \end{array} \right.$$ 

Consequently the sum is given by:

$$f(x) = \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{(2k-1)\pi} \cos \left( \frac{\pi x(2k-1)}{4} \right)$$

You could also obtain this formula from the formula from entry 6 (or others) by the correct transformations and using trig identities.
10. Consider the function \( f(x) = lx - x^2 \). Compute the sine series on \([0, l]\). One could obtain this formula using the correct transformations from the table but I will proceed directly, but will not carry out all the details of the integration to compute the integral for the \( b_n \). The formula for this is:

\[
b_n = 2 \left( \frac{l}{n^3 \pi^3} \left( \frac{1}{l} \right) \left( l^2 - ln^2 \frac{\pi x}{l} + n^2 \frac{\pi^2 x^2}{l} \right) \cos \left( \frac{n \pi x}{l} \right) + (ln \pi - 2 \pi n x \sin \left( \frac{n \pi x}{l} \right)) \right) \bigg|_0^l
\]

Thus the Fourier sine series is given by:

\[
lx - x^2 = \sum_{k=1}^{\infty} \frac{8l^2}{(2k - 1)^3 \pi^3} \sin \left( \frac{\pi(2k - 1)x}{l} \right) \quad (0 < x < l)
\]

12. Suppose that \( f \) is a piecewise continuous function on \([0, \pi]\) such that \( f(\theta) = f(\pi - \theta) \). Let \( a_n \) and \( b_n \) be the Fourier Sine and Cosine coefficients of \( f \). Show that \( a_n = 0 \) for \( n \) odd and that \( b_n = 0 \) for \( n \) even. Here I will only show the calculation for \( a_n \) but the calculation for \( b_n \) is similar.

The formula for \( a_n \) for a general \( f \) is given by:

\[
a_n = \frac{2}{\pi} \int_0^\pi f(\theta) \cos(n\theta) d\theta
\]

By splitting the integral in 2 and then using the fact that \( f(\theta) = f(\pi - \theta) \) on the first integral we obtain:

\[
a_n = \frac{2}{\pi} \int_0^{\pi/2} f(\theta) \cos(n\theta) d\theta + \frac{2}{\pi} \int_{\pi/2}^{\pi} f(\theta) \cos(n\theta) d\theta
\]

Now we do a change of variables where we let \( \omega = \pi - \theta \) in the second integral.

\[
a_n = \frac{2}{\pi} \int_0^{\pi/2} f(\theta) \cos(n\theta) d\theta - \frac{2}{\pi} \int_{\pi/2}^{\pi/2} f(\omega) \cos(n(\pi - \omega)) d\omega
\]

Now I use the fact that:

\[
\cos(n(\pi - \omega)) = \cos(n\omega) \cos(n\pi) + \sin(n\pi) \sin(n\omega) = (-1)^n \cos(n\omega)
\]

Substituting this into the equation above we get:

\[
a_n = \frac{2}{\pi} \int_0^{\pi/2} f(\theta) \cos(n\theta) d\theta + (-1)^n \frac{2}{\pi} \int_{\pi/2}^{\pi/2} f(\omega) \cos(n\omega) d\omega
\]

This when \( n \) is odd we have that:

\[
a_n = \frac{2}{\pi} \int_0^{\pi/2} f(\theta) \cos(n\theta) d\theta - \frac{2}{\pi} \int_{\pi/2}^{\pi/2} f(\omega) \cos(n\omega) d\omega = 0
\]
Section 2.5

1. A rod 100 cm long is insulated along its length and at both ends. Suppose that its initial temperature is \( u(x, 0) = x \) (\( x \) in cm, \( u \) in degrees C, \( t \) in sec, \( 0 \leq x \leq 100 \)), and that its diffusivity coefficient \( k \) is 1.1 cm\(^2\)/sec (about right if the bar is made of copper).

(a) Find the temperature \( u(x, t) \) for \( t > 0 \). (It is something of the form \( 50 + \sum_{n=1}^{\infty} a_n(t) \cos(n\pi x/100) \), and \( a_n(t) = 0 \) when \( n \) is even.)

The general series solution for this problem is found by separation of variables and is carried out on pages 14 and 15 in the text. The formula that results is the second equation on page 15 with \( l = 100 \) and \( k = 1.1 \), Namely:

\[
 u(x, t) = \sum_{n=0}^{\infty} a_n e^{-\frac{n^2 \pi^2}{100^2} t} \cos(n \pi x/100)
\]

Evaluating this at \( t = 0 \) we obtain that we must have for \( 0 \leq x \leq 100 \),

\[
 x = u(x, 0) = \sum_{n=0}^{\infty} a_n \cos(n \pi x/100)
\]

Thus the \( a_n \) are the Fourier cosine coefficients for the function \( x \) on the interval \([0, 100]\). This can be done directly computing the integral:

\[
 a_n = \frac{2}{100} \int_{0}^{100} x \cos(n \pi x/100) dx,
\]

or one can look at the fourier series for \( \theta \) (the even extension of \( \theta \)). This says that:

\[
 |\theta| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{1}^{\infty} \cos((2n-1)\pi x/100) \frac{\theta}{(2n-1)^2} \quad (-\pi < \theta < \pi)
\]

If you replace \( \theta = \pi x/100 \) one obtains:

\[
 \frac{\pi x}{100} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{1}^{\infty} \cos((2n-1)\pi x/100) \frac{1}{(2n-1)^2} \quad (-100 < x < 100)
\]

Multiplying both sides by \( 100/\pi \) we obtain:

\[
 |x| = 50 - \frac{400}{\pi^2} \sum_{1}^{\infty} \cos((2n-1)\pi x/100) \frac{1}{(2n-1)^2} \quad (-100 < x < 100)
\]

This is exactly the Fourier Cosine Series for \( x \) on the interval \([0, 100]\), and gives us the \( a_n \) coefficients. Replacing this into the formula for \( u \) gives us:

\[
 u(x, t) = 50 - \frac{400}{\pi^2} \sum_{1}^{\infty} e^{-\frac{(2n-1)^2 \pi^2}{100^2} t} \cos((2n-1)\pi x/100) \frac{1}{(2n-1)^2} \quad (x < 100)
\]

(b) Pulling off the first three terms we could write:

\[
 u(x, t) = 50 - \frac{400}{\pi^2} e^{-\frac{(1)^2 \pi^2}{100^2} t} \cos(\pi x/100) - \frac{400}{\pi^2} e^{-\frac{9(1)^2 \pi^2}{100^2} t} \cos(9 \pi x/100) + E(x, t),
\]
where,

\[ E(x, 60) = -\frac{400}{\pi^2} \sum_{n=3}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{100^2}} \cos \left(\frac{(2n-1)\pi x}{100}\right) \frac{1}{(2n-1)^2} \]

The claim of this problem is that when \( t = 60 \) that \( |E(x, 60)| < 1 \) for all \( x \). To show this we invoke the triangle inequality to say:

\[
|E(x, 60)| = \left| \frac{400}{\pi^2} \sum_{n=3}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{100^2}} \cos \left(\frac{(2n-1)\pi x}{100}\right) \frac{1}{(2n-1)^2} \right| 
\leq \frac{400}{\pi^2} \sum_{n=3}^{\infty} \left| e^{-\frac{(2n-1)^2\pi^2}{100^2}} \cos \left(\frac{(2n-1)\pi x}{100}\right) \right| \frac{1}{(2n-1)^2} 
\leq \frac{400}{\pi^2} \sum_{n=3}^{\infty} \left| e^{-\frac{(2n-1)^2\pi^2}{100^2}} \right| \frac{1}{(2n-1)^2} 
\leq \frac{400}{\pi^2} \sum_{n=3}^{\infty} \left| e^{-\frac{(2n-1)^2\pi^2}{100^2}} \right| \frac{1}{(2n-1)^2} 
\leq \frac{400}{\pi^2} \sum_{n=3}^{\infty} \left| e^{-\frac{(2n-1)^2\pi^2}{100^2}} \right| \frac{1}{(2n-1)^2} 
\approx 0.974925 < 1 
\]

The last equality is obtained using the formula written in the book. Now we also can then write

\[
\begin{align*}
u(0, 60) &= 9.52179 \pm 1 \\
u(10, 60) &= 12.4131 \pm 1 \\
u(40, 60) &= 40.2929 \pm 1 
\end{align*}
\]

(c) Find a number \( T > 0 \) such that \( u(x, t) \) is within 1 unit of its equilibrium value 50 for all \( x \) when \( t > T \). The argument for this is similar to the one used above to bound \( E(x, 60) \). We again use the triangle inequality and the fact that cosine is bounded in absolute value by 1 to get:

\[
|u(x, t) - 50| = \left| \frac{400}{\pi^2} \sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{100^2}} \cos \left(\frac{(2n-1)\pi x}{100}\right) \frac{1}{(2n-1)^2} \right| 
\leq \frac{400}{\pi^2} \sum_{n=1}^{\infty} \left| e^{-\frac{(2n-1)^2\pi^2}{100^2}} \cos \left(\frac{(2n-1)\pi x}{100}\right) \right| \frac{1}{(2n-1)^2} 
\leq \frac{400}{\pi^2} \sum_{n=1}^{\infty} \left| e^{-\frac{(2n-1)^2\pi^2}{100^2}} \right| \frac{1}{(2n-1)^2} 
\leq \frac{400}{\pi^2} \sum_{n=1}^{\infty} \left| e^{-\frac{(2n-1)^2\pi^2}{100^2}} \right| \frac{1}{(2n-1)^2} 
\leq \frac{400}{\pi^2} \sum_{n=1}^{\infty} \left| e^{-\frac{(2n-1)^2\pi^2}{100^2}} \right| \frac{1}{(2n-1)^2} 
\approx 0.974925 < 1 
\]

Now for any positive \( t \) the exponential has its largest value at the smallest value of \( n \) which is 1.
in this case, consequently $e^{-\frac{(2n-1)^2+5^2}{100^2}} < e^{-\frac{2(1.1)^2}{100^2}}$ for all $n > 1$. Thus:

$$|u(x, t) - 50| \leq \frac{400e^{-\frac{2(1.1)^2}{100^2}}}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n-1)^2} \leq \frac{400e^{-\frac{2(1.1)^2}{100^2}}}{8}$$

The last equality was obtained by the Hint given in the text. Let $T = \frac{100^2 \ln(400/8)}{(1.1)\pi^2}$. Then if $t > T$ we have that:

$$t > \frac{100^2 \ln(400/8)}{(1.1)\pi^2}$$

$$\frac{(1.1)^2 \pi^2 t}{100^2} > \ln(400/8) = -\ln(8/400)$$

$$-\frac{(1.1)^2 \pi^2 t}{100^2} < \ln(8/400)$$

$$e^{-\frac{(1.1)^2 \pi^2 t}{100^2}} < 8/400$$

$$\frac{400e^{-\frac{(1.1)^2 \pi^2 t}{100^2}}}{8} < 1$$

Thus if $t > T$ we have that $|u(x, t) - 50| < 1$. What this says is that after $T$ seconds the temperature will be within 1 of 50 everywhere on the rod.

3. (a) For this problem we can look back at section 1.3 to see how to start this problem using separation of variables. The solution that they arrive at is given in formula (2.25). The goal is then to find the coefficients $b_n$ such that:

$$x = \sum_{1}^{\infty} b_n \sin(\pi n x/100)$$

(Here $l$ is 100). This means we need the Fourier sine series of $x$ on the interval $[0, 1]$. This can be obtained directly or by scaling and multiplying entry 1 of the table. Doing this gives that:

$$x = \frac{200}{\pi} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(\pi n x/100), \quad (-100 < x < 100)$$

Once we have this we can assemble the series solution given by:

$$u(x, t) = \frac{200}{\pi} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{n^2 \pi^2 t}{100^2}} \sin(\pi n x/100)$$

(b) Numerically I computed these series using Mathematica and got:

$$u(50, 30) = 50 \pm 1 \quad (8 \text{ terms})$$
$$u(50, 60) = 49.96 \pm 1 \quad (8 \text{ terms})$$
$$u(50, 300) = 44.84 \pm 1 \quad (8 \text{ terms})$$
$$u(50, 3600) = 1.28 \pm 1 \quad (8 \text{ terms})$$
(c) to prove this face we note that \( u(50, t) \) is given by:

\[
 u(50, t) = \frac{200}{\pi} \sum_{1}^{\infty} \frac{1}{2n - 1} \frac{e^{-(2n-1)^2 \pi^2 t^{2} (1/100)^2}}{(2n-1)^{10}} (-1)^{n+1}
\]

For any \( t \), the exponential term is oscillating. This means that the resulting series is oscillating. A fact about oscillating series is that the remainder of the series is bounded by the absolute value of the next term. In other words if I use the first 8 terms of the series to get the answers in (b) then they are at most off by:

\[
\left| \frac{200}{\pi} \frac{e^{-(9)^2 \pi^2 t^{2} (1/100)^2}}{(-1)^{10}} \right|
\]

Note that I have used the value \( n = 5 \) because that corresponds to \( 2n - 1 = 9 \) in the original series. This value is a decreasing function of \( t \) and thus for the values of \( t \) considered above it will be largest at \( t = 30 \). Plugging in the value and checking numerically we have that:

\[
\left| \frac{200}{\pi} \frac{e^{-(9)^2 \pi^2 t^{2} (1/100)^2}}{(-1)^{10}} \right| \approx 0.506
\]

Thus the error in all the above calculations is no more then this number which is less then 1.

4. Consider the vibrating string occupying the interval \( 0 \leq x \leq l \). Suppose the string is plucked in the middle in such a way that its initial displacement \( u(x, 0) \) is \( 2m x/l \) for \( 0 \leq x \leq l/2 \) and \( 2m(l - x)/l \) for \( l/2 \leq x \leq l \) (so the maximum displacement, at \( x = l/2 \) is \( m \)) and its initial velocity \( u_t(x, 0) \) is zero.

(a) Find the displacement \( u(x, t) \) as a Fourier Series: Based on the calculations in section 1.3 as well as 2.5 we have that the solution for this problem is given by:

\[
 u(x, t) = \sum_{1}^{\infty} \sin(n\pi x/l) \left( b_n \cos \left( \frac{n\pi ct}{l} \right) + \frac{l B_n}{n\pi c} \sin \left( \frac{n\pi ct}{l} \right) \right)
\]

\[
 f(x) = \begin{cases} 
 2mx/l & 0 \leq x \leq l/2 \\
 2m(l-x)/l & l/2 \leq x \leq l 
\end{cases} = \sum_{1}^{\infty} b_n \sin(n\pi x/l) \quad 0 = \sum_{1}^{\infty} B_n \sin \left( \frac{n\pi x}{l} \right)
\]

Consequently, the \( B_n \) are all 0 and the \( b_n \) can be determined by computing the Fourier Cosine series for \( f(x) \). The \( b_n \) are then given by:

\[
 b_n = \frac{2}{l} \int_{0}^{l} f(x) \cos(n\pi x/l) dx = \frac{2}{l} \int_{0}^{l/2} 2mx/l \cos(n\pi x/l) + \frac{2}{l} \int_{l/2}^{l} 2m(l-x)/l \cos(n\pi x/l) dx = \begin{cases} 
 0 & n \text{ even} \\
 \frac{(-1)^{k+1} 8m}{(2k-1)^2 \pi l} & n = 2k - 1 
\end{cases}
\]

I have skipped the steps for carrying out this integration and I leave that to you.

(b) To show that this is of the form suggested we can write the solution as:

\[
 u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)]
\]

where \( f \) is the odd periodic extension of \( f \) defined above. This is because \( g \) is a 0 so the antiderivative \( G \) of \( g \) is a constant and this \( G(x - ct) - G(x + ct) = 0 \) in the formula in (2.28). Now at \( x = 0 \) we note that \( f(-ct) + f(ct) = 0 \) since \( f \) is extended as an odd function. Now there are a few cases, I will only consider one case here. Suppose that \( l/2 + 2lk < ct < l + 2lk \) for some integer \( k \), then \(-l - 2lk < ct < -l/2 - 2lk\). In both of these cases \( f \) is decreasing with a constant slope, thus initially for fixed \( t \) and as a function of \( x \), \( u(x, t) \) has a constant negative slope of \(-2n/l\). This continues for \( 0 < x < ct - l/2 - 2lk \). When \( l/2 - 2lk < x - ct < 2lk - 1 \) then
\( f(x - ct) \) begins to increase with slope \(-2m/l\) but \( f(x + ct) \) remains decreasing. Since the slopes are opposite this means \( u(x, t) \) is constant for this interval until \( f(x + ct) \) begins to increase as well. Thus having the second shape on page 57. The other cases are similar. This is also easier to explain graphically which I omit here.

Section 2.6

1. (a) Given that:

\[
g_n(\theta) = 2 \sum_{1}^{N} \frac{\sin(n \theta)}{n} - (\pi - \theta)
\]

we can differentiate term by term (there is no problem with this since it is a finite series). This gives that:

\[
g_n'(\theta) = 2 \sum_{1}^{N} \frac{\cos(n \theta)}{n} + 1 = 1 + 2 \sum_{1}^{N} \frac{e^{in\theta} + e^{-in\theta}}{2} = 1 + \sum_{1}^{N} e^{in\theta} + \sum_{1}^{N} e^{-in\theta} = 2\pi D_N.
\]

Here I have used the formula for \( D_N \) given on line (2.12). I have also used the Euler Formula for \( \cos(n \theta) \) as well as reindexing to combine the sums in the second to last inequality.

(b) To prove the facts in (b) I use the formulation of \( D_N \) given in (2.14) that says:

\[
D_N(\theta) = \frac{\sin((N + 1/2)\theta)}{\sin(\theta/2)}.
\]

Since \( g_n'(\theta) = 2\pi D_N(\theta) \) then the critical points of \( g \) correspond to values of \( \theta \) where \( D_N = 0 \).

This means that \( \sin((N + 1/2)\theta) = 0 \), or \( \theta = \pi(N + 1/2)k \) for some integer \( k \). The first value of \( k \) such that \( \theta > 0 \) is at \( k = 1 \) and we call this value \( \theta_N = \pi(N + \frac{1}{2}) \). Also, since \( g_n'(\theta) = 2\pi D_N \) we know that:

\[
g_N(\theta) = \int_{0}^{\theta} D_N(\phi)d\phi + c.
\]

Evaluating this expression at 0 tells us that \( g_N(0) = c \) and using the definition of \( g_N \) above we can conclude that \( c = -\pi \). Now replacing \( D_N \) with its definition in (2.14) and putting in \( c \) we have that:

\[
g_N(\theta) = \int_{0}^{\theta} \frac{\sin((N + 1/2)\phi)}{\sin(\phi/2)}d\phi - \pi.
\]

Evaluating at \( \theta = \theta_N \) gives us that:

\[
g_N(\theta_N) = \int_{0}^{\theta_N} \frac{\sin((N + 1/2)\phi)}{\sin(\phi/2)}d\phi - \pi.
\]

(c.) Now to evaluate the limit of \( g_N(\theta_N) \) we first manipulate the integral in the expression above using the substitution in the hint. where we replace \((N + 1/2)\phi = \psi\) This gives that:

\[
\int_{0}^{\pi/(N+1/2)} \frac{\sin((N + 1/2)\phi)}{\sin(\phi/2)}d\phi - \pi = \int_{0}^{\pi} \frac{\sin(\psi)}{(N + 1/2)\sin((\psi/2)/(N + 1/2))}d\psi
\]

We now want to take a limit of this function (note this is better then the limit of the original statement because there are no \( N \)'s in the integrand. We also must use a fact to bring the limit inside the integrand, but this is a detail that we will not worry ourselves with right now. Consequently we arrive at:

\[
\lim_{N \to \infty} g_N(\theta_N) = \int_{0}^{\pi} \frac{\sin(\psi)}{\lim_{N \to \infty} [(N + 1/2)\sin((\psi/2)/(N + 1/2))]}d\psi - \pi
\]
Now we proceed to evaluate the limit
\[
\lim_{N \to \infty} \left( N + \frac{1}{2} \right) \sin \left( \frac{\psi}{N + \frac{1}{2}} \right) = \lim_{N \to \infty} \frac{\sin \left( \frac{\psi}{N + \frac{1}{2}} \right)}{\frac{1}{N + \frac{1}{2}}} = \lim_{N \to \infty} \frac{-\frac{\psi}{2} (N + 1/2)^{-2} \cos \left( \frac{\psi}{N + \frac{1}{2}} \right)}{-(N + 1/2)^{-2}} = \frac{\psi}{2}.
\]

In the second to last inequality above I have used L’Hôpital’s Rule. Inserting this back into our expression above we obtain:

\[
\lim_{N \to \infty} g_N(\theta_N) = \int_{0}^{\pi} \frac{\sin(\psi)}{\psi/2} d\psi - \pi.
\]

This is what we set out to prove. Now as the problem states this integral minus \( \pi \) is a nonzero number. What is this telling you? Its saying that difference the truncated Fourier series for \( \pi - \theta \) and the function it self has a maximum at \( \theta_N \). Then the limit of this difference approaches a nonzero number as \( N \to \infty \). This means that the Fourier series will "over shoot" by this much. This is the Gibbs Phenomenon and what we have observed happening in many problems when dealing with jump discontinuities.