Problem 1. Two types of coins are produced at a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas, if it lands heads less than 525 times then we shall conclude that it is the fair coin. If the coin is actually fair what is the probability that we shall reach a false conclusion? What would it be if the coin were biased? State clearly any assumptions you use?

Solution: Let $X$ be the number of heads that appear on 1000 flips. If I am flipping the fair coin this is a binomial random variable with $n = 1000$ and $p = .5$. If I am flipping the biased coin then it is binomial $n = 1000$ $p = .5$ If we are in either case, then we can approximate using the normal approximation. For the first problem you are interested in:

$$P\{X \geq 525\mid \text{fair coin is tossed}\} \approx P\left\{\frac{X - 500}{\sqrt{250}} \geq 1.5495\right\} = 1 - \Phi(1.5495)$$

$$1 - .9394 = .0606$$

For the second part we do a similar computation:

$$P\{X < 525\mid \text{biased coin is tossed}\} \approx \Phi(-1.6209) = .0526$$

Problem 2. One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that number 6 will appear between 150 and 200 times inclusively. If number 6 appears exactly 200 times, find the probability that number 5 will appear less than 150 times. State clearly any assumptions.
Solution: Here again for the first part we let $X$ be the number of 6’s that appear on 1000 rolls, then $X$ is binomial with $n = 1000$ and $p = 1/6$. We are interested in $P(150 < X < 200)$ and if we let $Z = (X - np)/(\sqrt{np(1-p)})$, then $Z$ can be approximated using the standard normal random variable thus we have that

$$P(150 < X < 200) \approx P(-1.3718 < Z < 2.7860) = \Phi(2.7860) - 1 + \Phi(1.3718) = .9121.$$  

Note: You would get a different number if you did $150 \leq X \leq 200$ which would also be a reasonable solution as well.

For the second part of the solution we just consider that we are in the reduced sample space in which we have 800 flips which must be 1, 2, 3, 4, 5. Thus if we let $Y$ be the number of 5’s that appear in this situation then $Y$ is binomial with $n = 800$ and $p = 1/5$, and we are interested in $P(Y < 150)$ and we can approximate by a uniform normal random variable $Z$. This gives:

$$P(Y < 150) \approx P(Z < -.9281) = 1 - \Phi(.9281) = .1762.$$  

Problem 3: Let $X$ be a continuous random variable having cumulative distribution function $F$. Define the random variable $Y$ bye $Y = F(x)$. Show that $Y$ is uniformly distributed over $(0, 1)$.

Solution: Here we first recognize that since $F$ is a distribution function then it only takes values between and possibly including 0 and 1. Thus the density function for $Y$ will be 0 for $y$ outside of $(0, 1)$. For $y \in (0, 1)$ then we note that $F$ is monotonic, thus the inverse exists and is increasing as well. Thus we have that:

$$F_Y(y) = P(F(X) \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y$$

Taking a derivative gives that $f_Y(y) = 1$ for $y \in (0, 1)$ thus:

$$f_Y(y) = \begin{cases} 
1 & y \in (0, 1) \\
0 & \text{otherwise}.
\end{cases}$$

Problem 4. Find the probability density function of $Y = e^X$ when $X$ is normally distributed with parameters $\mu$ and $\sigma^2$. (The random variable $Y$ is said to have a lognormal distribution since $\log Y$ has a normal distribution).

Solution: To compute the density function we can notice that $e^x$ is monotone increasing, and the inverse for $y = e^x$ is given by $ln(y)$. Thus applying the the theorem from class we have that:

$$f_Y(y) = \begin{cases} 
\frac{1}{y} e^{-\frac{(\ln(y) - \mu)^2}{2\sigma^2}} & y > 0 \\
0 & \text{otherwise}.
\end{cases}$$
Problem 5. The mean of a continuous random variable having distribution function $F$ is that value $m$ such that $F(m) = \frac{1}{2}$. That is, a random variable is just as likely to be larger than its median as it is to be smaller. Find the median of $X$ if $X$ is:

(a) uniformly distributed over $(a, b)$;

Solution: We set up:

$$\frac{1}{2} = \int_a^k \frac{1}{b-a} \, dx = \frac{k-a}{b-a}$$

Solving for $k$ gives $k = a + \frac{1}{2}(b-a)$

(b) normal with parameters $\mu$ and $\sigma^2$;

Solution: Here it is a straightforward change of variables shows that:

$$\frac{1}{2} = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx,$$

Thus the median is $\mu$

(c) exponential with rate $\lambda$.

We again doing the following:

$$\frac{1}{2} = \int_k^{\infty} \lambda e^{-\lambda x} \, dx = e^{-\lambda k}$$

Solving for $k$ gives that $k = \ln(2)/\lambda$.

Problem 6. The annual rainfall (in inches) in a certain region is normally distributed with expected value equal to 40 and standard deviation of 4. What is the probability that starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches? What assumptions are you making?

Solution: The first thing to do is to compute the probability that in any one year there is more then 50 inches of rain. If we let $X$ be the amount of rain in a given year then as given, $X$ is normal with $\mu = 40$ and $\sigma = 4$. We seek $P\{X > 50\}$:

$$P\{X > 50\} = P\{X - 40 > 2.5\} = 1 - \Phi(2.5) = 1 - .9938 = .0062$$

Now if $Y$ is the number of years until we have such a year, then $Y$ is a geometric r.v. where $p = .0062$. And consequently we have that $P\{Y > 10\} = (1 - p)^{10} = .9397$. You could also get to this answer by computing a sum as well.

Problem 7. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x} \quad x \geq 0.$$
Compute the expected lifetime of such a tube.

**Solution:** To do this you must compute the integral:

\[
E[X] = \int_0^\infty x^2 e^{-x} \, dx
\]

\[
= -x^2 e^{-x}\bigg|_0^\infty + \int_2^\infty 2x e^{-x} \, dx
\]

\[
= -2x e^{-x}\bigg|_0^\infty + 2 \int_0^\infty e^{-x} \, dx
\]

\[
= -2 e^{-x}\bigg|_0^\infty = 2
\]

**Problem 8.** If \( X \) is uniformly distributed over \((-1, 1)\), find

(a) \( P\{|X| > \frac{1}{2}\}\).

**Solution:**

\[
P\{|X| > \frac{1}{2}\} = P\{X > \frac{1}{2}\} + P\{X < -\frac{1}{2}\} = \int_{\frac{1}{2}}^1 \frac{1}{2} \, dx + \int_{-1}^{-\frac{1}{2}} \frac{1}{2} \, dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]

(b) the density function of the random variable \(|X|\).

**Solution:** To compute the density function we start with the distribution function:

\[
F_{|X|}(a) = P\{|X| \leq a\} = P\{-a \leq X \leq a\} = F_X(a) - F_X(-a)
\]

Now we differentiate and get that as long as \(0 < a < 1\):

\[
f_{|X|} = f_X(a) + f_X(-a) = \frac{1}{2} + \frac{1}{2} = 1.
\]

Thus \(|X|\) is a uniform random variable on \((0, 1)\).

**Problem 9.** The number of years a radio functions is exponentially distributed with parameter \(\lambda = \frac{1}{8}\). If Jones buys a used radio, what is the probability that it will be working after an additional 8 years? Explain all reasoning.
**Solution:** First let $X$ be the number of years that a particular radio is functioning. Since it behaves like an exponential random variable then it is memoryless. So the probability that a used radio will last an additional 8 years is the same as if computing the prob that $X > 8$. So,

$$P\{X > 8\} = \int_{8}^{\infty} \frac{1}{8} e^{-\frac{x}{8}} dx = e^{-1} \approx .3678.$$