Stability of plane-wave solutions to a dissipative generalization of the NLS equation

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Outline

1. Review of NLS results
2. A dissipative NLS equation
3. Stability of plane-wave solutions to DNLS
4. Comments on vector DNLS
The \((1+1)\)-dimensional cubic nonlinear Schrödinger equation is

\[ i\psi_t + \alpha \psi_{xx} + \gamma |\psi|^2 \psi = 0 \]

- \(\psi = \psi(x, t)\) is a complex-valued function
- \(\alpha\) and \(\gamma\) are real constants

The class of plane-wave solutions is given by

\[ \psi(x, t) = \psi_0 e^{ikx + i(-\alpha k^2 + \gamma \psi_0^2) t + i \xi} \]

- \(\psi_0, k\) and \(\xi\) are real constants

*These solutions have constant magnitude.*
Consider perturbed plane-wave solutions of the form

$$
\psi_p(x, t) = (\psi_0 + \epsilon u(x, t) + i\epsilon v(x, t)) e^{ikx + i(-\alpha k^2 + \gamma \psi_0^2) t + i\xi}
$$

where

- $\epsilon$ is a small real parameter
- $u(x, t)$ and $v(x, t)$ are real-valued functions
WOLOG assume

\[ u(x, t) = \overline{U} e^{ipx + \Omega t} + c.c. \]

\[ v(x, t) = \overline{V} e^{ipx + \Omega t} + c.c. \]

where
- \( \overline{U} \), \( \overline{V} \) and \( \Omega \) are complex constants
- \( p \) is a real constant
- \( c.c. \) denotes complex conjugate
This leads to the following system of algebraic equations

\[
\begin{pmatrix}
2\gamma\psi_0^2 - \alpha p^2 & -\Omega - 2i\alpha kp \\
\Omega + 2i\alpha kp & -\alpha p^2
\end{pmatrix}
\begin{pmatrix}
U \\
V
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

In order for this system to have a nonzero solution, \( \Omega \) must be chosen to satisfy

\[
\Omega_{\pm} = -2i\alpha kp \pm \sqrt{\alpha p^2(2\gamma\psi_0^2 - \alpha p^2)}
\]
NLS Stability Observations

\[ \Omega_{\pm} = -2i\alpha kp \pm \sqrt{\alpha p^2 (2\gamma \psi_0^2 - \alpha p^2)} \]

- If \( \alpha \gamma \leq 0 \), then there is no instability.
- If \( \alpha \gamma > 0 \), then there is instability.
- The maximum growth rate is \( |\gamma| \psi_0^2 \).
- Note: \( k \) only contributes to the imaginary part of \( \Omega \).
A dissipative generalization of NLS (DNLS) is

\[ i\psi_t + (\alpha - ia)\psi_{xx} + (\gamma + ic)|\psi|^2\psi + id\psi = 0 \]

- \( \psi = \psi(x, t) \) is a complex-valued function
- \( \alpha \) and \( \gamma \) are real constants
- \( a, c \) and \( d \) are nonnegative real constants
The DNLS equation has arisen in many studies of water waves with dissipation

- Davey (1972) provides an argument for DNLS-like terms
- Lake et al. (1977) add the $id\psi$ term
- Blennerhassett (1980) derives an equation similar to DNLS
- Segur et al. (2005) add the $id\psi$ term
- Bridges & Dias (2007) study DNLS in a different manner
Unlike NLS, DNLS does not (in general) conserve the $L_2$-norm. Specifically, the $L_2$-norm is nonincreasing in $t$ and

$$\frac{d}{dt} \left( \int_{0}^{L_x} |\psi|^2 dx \right) = -2 \int_{0}^{L_x} \left( a|\psi_x|^2 + c|\psi|^4 + d|\psi|^2 \right) dx$$

Note that if $a > 0$, then solutions with spatial dependence decay more rapidly than spatially-independent solutions.
DNLS Plane-Wave Solutions

DNLS admits four classes of solutions of the form

$$\psi(x, t) = \psi_0 \exp(ikx + \omega_r(t) + i\omega_i(t) + i\xi)$$

where

- $\psi_0$, $k$ and $\xi$ are real constants
- $\omega_r(t)$ and $\omega_i(t)$ are real-valued functions
DNLS admits four classes of solutions of the form

\[ \psi(x, t) = \psi_0 \exp(ikx + \omega_r(t) + i\omega_i(t) + i\xi) \]

where

- \( \psi_0, k \) and \( \xi \) are real constants
- \( \omega_r(t) \) and \( \omega_i(t) \) are real-valued functions

Case #1: \( c = 0 \) and \( ak^2 + d = 0 \)

\[ \omega_r(t) = 0, \]
\[ \omega_i(t) = -t(\alpha k^2 - \gamma \psi_0^2). \]

*These solutions have constant magnitude.*
DNLS Plane-Wave Solutions: Case #2

\[ \psi(x, t) = \psi_0 \exp(ikx + \omega_r(t) + i\omega_i(t) + i\xi) \]

Case #2: \( c > 0 \) and \( ak^2 + d = 0 \)

\[ \omega_r(t) = -\frac{1}{2} \ln(1 + 2c\psi_0^2 t), \]

\[ \omega_i(t) = -\alpha k^2 t - \frac{\gamma}{c} \omega_r(t). \]

*The magnitude of these solutions decays like \( t^{-1/2} \).*
DNLS Plane-Wave Solutions: Case #3

\[
\psi(x, t) = \psi_0 \exp \left( ikx + \omega_r(t) + i\omega_i(t) + i\xi \right)
\]

Case #3: \( c = 0 \) and \( ak^2 + d > 0 \)

\[
\omega_r(t) = -t(ak^2 + d),
\]

\[
\omega_i(t) = -\alpha k^2 t + \frac{\gamma \psi_0^2}{2(ak^2 + d)} (1 - e^{-2t(ak^2+d)}).
\]

*The magnitude of these solutions decays exponentially.*
$\psi(x, t) = \psi_0 \exp(ikx + \omega_r(t) + i\omega_i(t) + i\xi)$

Case #4: $c > 0$ and $ak^2 + d > 0$

$$\omega_r(t) = \frac{1}{2} \ln \left( \frac{ak^2 + d}{(ak^2 + d + c\psi_0^2)e^{2t(ak^2+d)} - c\psi_0^2} \right),$$

$$\omega_i(t) = -t(\alpha k^2 + \frac{\gamma}{c}(ak + d)) - \frac{\gamma}{c}\omega_r(t).$$

*The magnitude of these solutions decays nearly exponentially.*
These four cases include all plane-wave solutions of DNLS.
Consider perturbed solutions of the form

\[ \psi_p(x, t) = (\psi_0 + \epsilon u(x, t) + i\epsilon v(x, t) + \mathcal{O}(\epsilon^2)) \exp(ikx + \omega_r(t) + i\omega_i(t) + i\xi) \]

where

- \( \epsilon \) is a small real parameter
- \( u(x, t) \) and \( v(x, t) \) are real-valued functions

Note that any decay due to \( \omega_r(t) \) has been factored out.
DNLS Stability Case #1 \( c = 0 \) and \( ak^2 + d = 0 \)

Similar to the NLS result

\[
\Omega_{\pm} = -ap^2 - 2i\alpha kp \pm \sqrt{\alpha p^2(2\gamma\psi_0^2 - \alpha p^2)}
\]

- \( \Re(\Omega_{\pm}) \) does not depend on \( k \).
- The maximal growth rate is less than or equal to the corresponding NLS maximal growth rate.
DNLS Stability Case #2 \[ c > 0 \text{ and } ak^2 + d = 0 \]
WOLOG Assume

\[ u(x, t) = U(t)e^{ipx} + c.c. \]
\[ v(x, t) = V(t)e^{ipx} + c.c. \]

where

- \( U(t), V(t) \) are complex-valued functions
- \( p \) are real constants
This leads to

\[
\begin{pmatrix} U \\ V \end{pmatrix}' = A \begin{pmatrix} U \\ V \end{pmatrix} + B(t) \begin{pmatrix} U \\ V \end{pmatrix}
\]

where

\[
A = \begin{pmatrix} -ap^2 - 2i\alpha kp & \alpha p^2 \\ -\alpha p^2 & -ap^2 - 2i\alpha kp \end{pmatrix}
\]

\[
B(t) = \frac{2\psi_0^2}{1 + 2c\psi_0^2 t} \begin{pmatrix} -c & 0 \\ \gamma & 0 \end{pmatrix}
\]

This system has a solution in terms of hypergeometric functions. These solutions are bounded for all \( t \) for all parameter choices.
DNLS Stability Cases #3, #4

\[ c \geq 0 \text{ and } ak^2 + d > 0 \]
Similar calculations lead to

\[
\begin{pmatrix} U \\ V \end{pmatrix}' = A \begin{pmatrix} U \\ V \end{pmatrix} + B(t) \begin{pmatrix} U \\ V \end{pmatrix}
\]

where

\[
A = \begin{pmatrix} -ap^2 - 2\alpha kp & \alpha p^2 - 2iakp \\ -\alpha p^2 + 2iakp & -ap^2 - 2i\alpha kp \end{pmatrix}
\]

\[
B(t) = 2\psi_0^2 e^{-2t(ak^2+d)} \begin{pmatrix} -c & 0 \\ \gamma & 0 \end{pmatrix}
\]

Since \( B(t) \) decays exponentially, the stability of solutions of these forms is determined by the eigenvalues of \( A \).
The eigenvalues of $A$ are

\[
\begin{align*}
\lambda_1 &= -ap(2k + p) - i\alpha p(2k + p) \\
\lambda_2 &= ap(2k - p) - i\alpha p(2k - p)
\end{align*}
\]

- If $a = 0$, then both $\lambda_1$ and $\lambda_2$ have zero real part. Thus, all corresponding solutions are neutrally stable.
- If $k = 0$, then both $\lambda_1$ and $\lambda_2$ have nonpositive real part. Thus, all corresponding solutions are linearly stable.
- If $a \neq 0$ and $k \neq 0$, then any choice of $p$ for which

\[
p^2 < -2kp \quad \text{or} \quad p^2 < 2kp
\]

leads to an eigenvalue with positive real part. This establishes that all spatially-dependent solutions in this case are linearly unstable if $a$ is positive.
The maximum growth rate is

$$\max_p \Re(\lambda_1, \lambda_2) = ak^2$$

This growth rate is achieved if $p = -k$.

This establishes that the most unstable mode is the $p = -k$-mode.
Recall

\[ \psi_p(x, t) = (\psi_0 + \epsilon u(x, t) + i\epsilon v(x, t) + \mathcal{O}(\epsilon^2)) \exp(ikx + \omega_r(t) + i\omega_i(t) + i\xi) \]

The linear theory picks out the mode with the slowest overall decay rate (the 0-mode).
If $c > 0$, then the plane-wave solutions should decay to zero because the $\mathcal{L}_2$-norm is not conserved.

However, the calculations establish

- The maximum growth rate of an instability is $ak^2$.
- The decay rate due to $\omega_r(t)$ is $-(ak^2 + d)$.

If $d = 0$, then the linear theory states that spatially-independent solutions do not decay if $c > 0$. Clearly, this is contrary to the fact that the $\mathcal{L}_2$-norm is not conserved when $c > 0$...........

Numerics help sort out the apparent contradiction.
Consider the initial condition (a perturbed plane-wave solution)

\[ \psi(x, y, 0) = (1 + 0.1e^{-2ix+5iy})e^{2ix-5iy} = e^{2ix-5iy} + 0.1 \]

(a) Results from numerical simulations of the linear ODEs.
(b) Results from numerical simulations of DNLS.
Summary

- All spatially-independent plane-wave solutions of DNLS are linearly stable.
- If \( a = 0 \), then all plane-wave solutions of DNLS are linearly stable.
- If \( a > 0 \), then all spatially-dependent plane-wave solutions of DNLS are linearly unstable.
- All of these results generalize to two and three dimensions.
- **Work in progress:** Generalizing these results to the vector DNLS equation.