Bidirectional Whitham Equations as Models of Waves on Shallow Water

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Abstract

Hammack & Segur [1] conducted a series of surface water-wave experiments in which the evolution of long waves of depression was measured and studied. This present work compares time series from these experiments with predictions from numerical simulations of the KdV, Serre, and several unidirectional and bidirectional Whitham-type equations. These comparisons show that accurate predictions come from models that contain an accurate reproduction of the Euler phase velocity, sufficient nonlinearity, and surface tension effects. The focus is to compare the quality of several bidirectional Whitham models. Most interestingly, the comparisons show that the bidirectional Whitham equations can provide predictions that are more accurate than the unidirectional models even though the experiments were essentially one dimensional.

1. Introduction

Hammack & Segur [1] performed a series of tightly-controlled laboratory water-wave experiments in a long, narrow tank with relatively shallow (10 cm) undisturbed water and a wave maker at one end. The wave maker was a rectangular, vertically-moving piston located on the bottom of the tank, adjacent to a rigid wall at the upstream end of the tank. Experiments were initialized by rapidly moving the piston downward a prescribed amount that varied between experiments. This downward motion lead to the creation of initially rectangular waves wholly below the still water level, occupying the entire width and 61 cm of the upstream end of the tank. The evolution of the wave train downstream from the wave maker was investigated. Time series were collected at five gauges located 61 cm ($x = 0$, the downstream edge of the piston), 561 cm ($x = 500$), 1,061 cm ($x = 1000$), 1,561 cm ($x = 1500$), and 2,061 cm ($x = 2000$) downstream. The tank was long enough that waves reflecting from the far end of the tank did not impact the time series collected. Among other things, Hammack & Segur showed that many analytic and asymptotic results obtained from the KdV equation compared favorably with measurements from the experiments.

In this paper, we focus on the two experiments presented in Figures 2 and 3 of [1], which we refer to as experiment #2 and experiment #3 respectively. The experiments were identical except for the magnitude of the piston displacement and hence initial wave amplitude. In experiment #2, the piston moved downward 1 cm, producing a downstream propagating wave with an initial amplitude of 0.5 cm. In experiment #3, the piston stroke and initial amplitude were 3 cm and 1.5 cm respectively. The time series from both experiments show leading triangular waves of depression followed by series of trailing wave groups.

The main goal of this paper is to compare and evaluate a number of generalized Whitham equations by comparing their predictions with the experimental time series. In doing this, we demonstrate that in order to most accurately model these experimental measurements, a model must include (i) an accurate reproduction of the Euler phase velocity, (ii) sufficient nonlinearity, and (iii) surface tension effects. Additionally, we show that bidirectional Whitham equations can perform better than unidirectional Whitham equations even though the experiment is essentially unidirectional.

This paper is organized as follows. The model equations and their properties are presented in Section 2. Comparisons between the experimental time series and data from numerical simulations of these equations are included in Section 3. A summary is contained in Section 4.

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Figure 1: Plots of scaled linear phase velocity versus scaled wave number for the Euler (with zero surface tension), KdV, and Serre equations.

2. Model Equations

The equations that describe the irrotational motion of an inviscid, incompressible, homogeneous fluid with a free surface are known as the Euler equations [2]. As the experiments of interest here were conducted in a long, narrow tank, we use two-dimensional models (i.e. models with one horizontal and one vertical dimension). The linear phase velocity for the Euler equations is given by

$$\left( c_E \right)^2 = \frac{(g + \tau k^2) \tanh(kh_0)}{k},$$

(1)

where $g$ represents the acceleration due to gravity, $h_0$ represents the mean depth of the fluid, $\tau$ represents the coefficient of surface tension, and $k$ represents the wave number of the linear wave. The squared term in equation (1) establishes that the Euler equations are bidirectional (waves of each wave number can travel toward both $x = -\infty$ and $x = \infty$ as $t$ increases) as opposed to unidirectional (waves of each wave number travel toward only $x = -\infty$ or $x = \infty$ as $t$ increases). Since the Euler equations are difficult to work with, it is common to introduce the dimensionless parameters

$$\delta = \frac{h_0}{\lambda_0}, \quad \epsilon = \frac{a_0}{h_0},$$

(2)

in order to derive asymptotic models that are less complicated. Here $\lambda_0$ is a typical wavelength and $a_0$ is a typical wave amplitude. The parameter $\epsilon$ is a measure of nonlinearity and the parameter $\delta$ is a measure of wavelength or shallowness.

2.1. The KdV Equation

The Korteweg-de Vries (KdV) equation can be derived from the Euler equations by assuming that $\delta^2 \sim \epsilon \ll 1$ and truncating at $O(\epsilon^3)$. In other words, the waves are assumed to have small amplitude and large wavelength. The KdV equation has been well studied mathematically (e.g. [3, 4, 5]) and experimentally (e.g. [6, 7, 8]). In dimensional form, the KdV equation is given by

$$\eta_t + \sqrt{gh_0} \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \frac{1}{6} h_0^2 \sqrt{gh_0} \eta_{xxx} = 0,$$

(3)

where $\eta = \eta(x, t)$ represents the displacement of the free surface from its undisturbed level. The phase velocity for KdV is

$$c_K = \sqrt{gh_0 \left( 1 - \frac{1}{6} (kh_0)^2 \right)}.$$

(4)

Figure 1 contains a plot of KdV’s phase velocity. The plot establishes that KdV only accurately approximates Euler’s phase velocity near $kh_0 = 0$ (i.e. in the long-wave limit). For a given $k$, there is a unique $c_K$, so KdV is a unidirectional model (even though $k$ such that $|k|h_0 < \sqrt{6}$ travel toward $x = \infty$ and $k$ such that $|k|h_0 > \sqrt{6}$ travel toward $x = -\infty$).
2.2. The Serre Equations

The first strongly nonlinear, weakly dispersive set of Boussinesq-type equations was derived by Serre [9, 10]. Several years later, Su & Gardner [11] and Green & Naghdi [12] re-derived these equations using different methods. Johnson [13] presents a rigorous, perturbation theory derivation of the system. In the literature, the equations are referred to as both the Serre equations and the Green & Naghdi equations.

They are obtained by depth-averaging the Euler system and truncating the resulting set of equations at $O(\delta^4)$ without making any assumptions on $\epsilon$. This “full nonlinearity” makes the Serre equations ideal for studying large-amplitude or nearly breaking waves in the near shore zone. The dimensional Serre equations are given by

$$h_t + (h\bar{u})_x = 0,$$

(5)

$$\bar{u}_t + \bar{u}\bar{u}_x + gh_x - \frac{1}{3h} \left(h^3 \left(\bar{u}_{xt} + \bar{u}\bar{u}_{xx} - (\bar{u}_x)^2\right)\right)_x = 0,$$

(6)

where $\bar{u} = \bar{u}(x, t)$ represents the depth-averaged horizontal velocity of the fluid and $h = h(x, t)$ represents the local fluid depth. Fluid depth is related to surface displacement via

$$h(x, t) = h_0 + \eta(x, t).$$

(7)

The Serre equations are bidirectional, dispersive, and have the following linearized phase velocity

$$(c_s)^2 = \frac{3gb_0}{3 + (kh_0)^2}.$$  

(8)

Figure 1 shows that the Serre equations do a better job of approximating the Euler phase velocity than KdV, but the Serre phase velocity is only accurate for $|k|h_0 \lesssim 1.5$.

2.3. The Whitham Equation

Neither KdV nor Serre accurately reproduces linearized phase velocity of the Euler equations for a wide range of $kh_0$ values. In order to address this issue, Whitham [14, 15] proposed a generalization of KdV that is now known as the Whitham equation for waves on shallow water. In dimensional form, it is given by

$$\eta_t + \sqrt{gh_0} K \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x = 0,$$

(9)

where $K$ is a (dimensionless) Fourier multiplier defined by the symbol

$$\hat{K} f(k) = \sqrt{\frac{(1 + \frac{\tau k^2}{g}) \tanh(kh_0)}{kh_0}} \hat{f}(k),$$

(10)

where $\tau$ represents the coefficient of surface tension. The linear phase velocity for the Whitham equation is

$$c_w = \sqrt{\frac{(g + \tau k^2) \tanh(kh_0)}{k}}.$$  

(11)

The Whitham equation provides a unidirectional reproduction of the Euler phase velocity. In the last decade, the Whitham equation has received significant attention in the mathematics community. Ehrnström & Kalisch [16] proved the existence of and computed periodic traveling-wave solutions to the Whitham equation. Sanford et al. [17] and Johnson & Hur [18] established that large-amplitude, periodic traveling-wave solutions to the Whitham equation are unstable while small-amplitude, periodic traveling-wave solutions are stable if the wavelength is long enough. Moldabayev et al. [19] presented a scaling regime in which the Whitham equation can be derived and compared its dynamics with those from other models including the Euler equations. Hur [20] proved that solutions to the Whitham equation will break provided that the initial condition is sufficiently asymmetric. Deconinck & Trichtchenko [21] proved that the unidirectional nature of the Whitham equation causes it to miss some of the instabilities of the Euler equations. To our knowledge, only Trillo et al. [22] compared Whitham predictions with measurements from laboratory experiments. They showed that the Whitham equation provides an accurate model for the evolution of initial waves of depression, especially when nonlinear plays a significant role.

In addition to equation (9), Whitham also proposed

$$\eta_t + \sqrt{gh_0} K \eta_x + 3 \left(\sqrt{g(h_0 + \eta)} - \sqrt{gh_0}\right) \eta_x = 0,$$

(12)
as a model for small-amplitude, long waves. We refer to this equation as the Sqrt Whitham equation. It provides the same unidirectional reproduction of the Euler phase velocity as does the Whitham equation. The first term in the \( \eta = 0 \) Taylor series expansion of the terms in parentheses is exactly the nonlinear term in the Whitham equation. This means that the Sqrt Whitham equation is a higher-order nonlinear equation than the Whitham equation.

2.4. Bidirectional Whitham Systems

Boussinesq-type models are obtained from the Euler equations in the long-wave, small-amplitude limit. There is a large number of Boussinesq-type systems (e.g. [23, 24, 25]). For example, the KdV equation is a unidirectional version of the Boussinesq system [2]. There are multiple “Whithamized” Boussinesq models, that is Boussinesq-type systems that have been modified so that their phase velocities match the bidirectional phase velocity of the Euler equations.

Aceves-Sánchez et al. [26] constructed the following Hamiltonian, bidirectional Whitham system

\[
\eta_t + h_0 K^2 u_x + (\eta u)_x = 0, \tag{13a}
\]
\[
u + g \eta_x + u u_x = 0, \tag{13b}
\]

where \( K \) is defined in equation (10). We refer to this system as the ASMP system. It has the following conserved quantities

\[
Q_1 = \int_0^L h \, dx, \tag{14a}
\]
\[
Q_2 = \int_0^L u \, dx, \tag{14b}
\]
\[
Q_3 = \int_0^L h u \, dx, \tag{14c}
\]
\[
Q_4 = \int_0^L \left( g \eta^2 + h_0 u K u + \eta u^2 \right) dx, \tag{14d}
\]

where \( L \) is the spatial period of the solution. The fourth conserved quantity, \( Q_4 \), is the Hamiltonian of the system. This system is bidirectional and has a linear phase velocity that exactly matches that of the Euler equations (i.e. it is “fully dispersive”). It is unknown whether or not this system is well posed, even for short times. Moldabayev et al. [19] presented a scaling regime in which the ASMP system can be derived from the Euler equations. Additionally, they extended their procedure and derived a fully dispersive Hamiltonian system with higher-order nonlinearity. This system is given by

\[
\eta_t + h_0 K^2 u_x + (\eta u)_x + \mu^2 h_0^2 (\eta u_x)_{xx} = 0, \tag{15a}
\]
\[
u + g \eta_x + u u_x - \mu^2 h_0^2 u_x u_{xx} = 0, \tag{15b}
\]

where \( \mu \) is a dimensionless parameter representing the ratio of \( h_0 \) and a typical wavelength. (In our numerical simulations, we used the wavelength of the initial surface displacement to define \( \mu \).) We refer to this system as the MKD system. It is bidirectional, fully dispersive and conserves \( Q_1, Q_2, Q_3 \), and its Hamiltonian,

\[
Q_5 = \int_0^L \left( g \eta^2 + h_0 u K u + \eta u^2 - \mu^2 h_0^2 (\eta u_x + \eta uu_{xx}) \right) dx. \tag{16}
\]

Another Whithamized Boussinesq system was proposed by Hur & Pandey [27]. In dimensional form, this system is given by

\[
\eta_t + h_0 u_x + (\eta u)_x = 0, \tag{17a}
\]
\[
u + g K^2 \eta_x + u u_x = 0. \tag{17b}
\]

We refer to this system as the HP system. Hur & Pandey established the system is well-posed for short times and that periodic traveling-wave solutions are spectrally unstable with respect to long-wave perturbations. The HP system conserves \( Q_1, Q_2, \) and \( Q_3 \), but not \( Q_4 \) or \( Q_5 \). It is not known whether or not the HP system is Hamiltonian. The HP system is equivalent to the ASMP system up to \( \mathcal{O}(\epsilon^3) \).
3. Numerics and Comparisons

3.1. Initial conditions

For simplicity, we ignored the motion of the piston, assumed that the bottom of the tank is horizontal, and assumed that the initial wave profile is a nearly rectangular wave of depression. Additionally, we assumed periodic boundary conditions on an interval large enough that waves did not wrap around and erroneously influence the numerical gauges. The initial conditions used for the numerical simulations of the KdV, Whitham, and Sqrt Whitham equations were

\[
\eta(x,0) = \begin{cases} 
0 & -7869 \leq x < -183, \\
-\frac{1}{2}A_0 + \frac{1}{2}A_0 \text{sn}(0.0925434x, 0.9999^2) & -183 \leq x \leq 61, \\
0 & 61 < x \leq 7747, 
\end{cases}
\] (18)

where \(A_0\) is the amplitude of the piston motion (\(A_0 = 0.5\) in experiment \#2 and \(A_0 = 1.5\) in experiment \#3) and \(\text{sn}(\cdot, m)\) is a Jacobi elliptic function with elliptic modulus \(m\) [28]. A plot of the nonzero portion of this initial condition is included in Figure 2. The position \(x = 0\) in the numerical tank corresponds to the rightmost/downstream edge of the experimental piston. The parameters in the initial conditions were chosen so that numerical gauges could be placed at \(x = 0\) and close to \(x = 500, 1000, 1500, 2000\) (the locations of the experimental gauges).

Due to the relation between \(h\) and \(\eta\) (see equation (7)), the initial conditions used for simulations of the Serre and bidirectional Whitham equations were

\[
h(x,0) = 10 + 2\eta(x,0),
\]

\[
\bar{u}(x,0) = u(x,0) = 0,
\] (19a)

where \(\eta(x,0)\) is defined in equation (18). The amplitude factor of 2 is necessary because these equations are bidirectional while the experiments, the KdV, and Whitham equations are unidirectional. Unfortunately the horizontal velocities were not measured in the experiments. Therefore, it is unclear how to properly choose them, especially since the fluid motion was initiated by a vertically moving piston. To avoid confusion, we chose to set the initial horizontal fluid velocities equal to zero. The impact of this choice (and others) is discussed in Section 3.2.4.

The Serre equations were solved using the iterative pseudospectral method presented by Dutykh et al. [29]. All other models were solved using sixth-order split-step [30] pseudospectral methods where the linear and nonlinear parts were solved independently. This allowed the linear parts of the PDEs (including the linear parts of the fully dispersive PDEs) to be solved exactly fast Fourier transforms. A three-eighths rule was required to solve the MKD system for experiment \#3, but was not used in any of the other simulations.

3.2. Results

In order to quantitatively compare the predictions from the various models, we used the norm

\[
\mathcal{E} = \sum_1^5 \frac{\sum_{j=1}^{N_k} |\text{expt}(k,j) - \text{num}(k,j)|^2}{\sum_{j=1}^{N_k} |\text{expt}(k,j)|^2},
\]

(20)

where \(N_k\) is the number of experimental temporal data points at gauge \(k\) (there were five gauges), and \(\text{expt}(k,j)\) and \(\text{num}(k,j)\) are the \(j^{th}\) experimental and numerical data values for the \(k^{th}\) gauge respectively. Table 1 contains \(\mathcal{E}\) values for all model equations and both experiments.
Table 1: Error values, see equation (20), for all model equations and both experiments based on the initial conditions given in equations (18) and (19).

<table>
<thead>
<tr>
<th>Model</th>
<th>$E_{\text{expt. } 2}$</th>
<th>$E_{\text{expt. } 3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PDE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>KdV</td>
<td>0.1238</td>
<td>1.6828</td>
</tr>
<tr>
<td>Linear Theory</td>
<td>0.6294</td>
<td>—</td>
</tr>
<tr>
<td>Serre</td>
<td>0.1048</td>
<td>1.4209</td>
</tr>
<tr>
<td>Whitham with $\tau = 0$</td>
<td>0.0734</td>
<td>0.9363</td>
</tr>
<tr>
<td>Whitham</td>
<td>0.0717</td>
<td>0.9212</td>
</tr>
<tr>
<td>Sqrt Whitham</td>
<td>0.0716</td>
<td>0.9740</td>
</tr>
<tr>
<td>ASMP with $\tau = 0$</td>
<td>0.0788</td>
<td>1.1090</td>
</tr>
<tr>
<td>ASMP</td>
<td>0.0777</td>
<td>1.1119</td>
</tr>
<tr>
<td>HP with $\tau = 0$</td>
<td>0.0798</td>
<td>1.2316</td>
</tr>
<tr>
<td>HP</td>
<td>0.0786</td>
<td>1.2056</td>
</tr>
<tr>
<td>MKD</td>
<td>0.0735</td>
<td>0.9799</td>
</tr>
</tbody>
</table>

3.2.1. The KdV Equation

Figure 3 contains plots that compare results from numerical simulations of KdV with the experimental data for experiment #2 at each of the five gauges. Figure 3(a) shows that, at the gauge closest to the wave maker, KdV does a reasonable job modeling the amplitudes of the initial wave of depression and first wave of elevation, but overestimates the amplitudes of the remaining trailing waves. Figures 3(b)-(e) include the results from the four downstream gauges. They show that KdV accurately predicts the amplitude of the initial wave of depression, but overestimates the amplitudes of the trailing waves of elevation. This amplitude overestimation may be due to experimental dissipative effects that KdV (a nondissipative model) does not account for.

At each gauge, KdV accurately models the phases of the first two waves (the initial wave of depression and the initial wave of elevation), but does not accurately model the phases of any of the trailing waves. As time increases at each gauge, this phase error increases. This is consistent with the fact that KdV most poorly models the speeds of the slowest waves. Finally, KdV completely misses the group nature of the trailing dispersive waves.

Table 1 shows that KdV provided the second least accurate prediction for the experimental data according to the error norm given in equation (20). The only model that KdV outperformed is the “linear theory” model. This model is simply the (bidirectional) linearized Euler equations. It does a poor job modeling the data from experiment #2 and does not even qualitatively predict the wave evolution in experiment #3 (and therefore its result is left out of Table 1). The poor performance of the linear theory suggests that some form of nonlinearity is required in order to accurately model data from these experiments. The role of nonlinearity is further discussed in Sections 3.2.2 and 3.2.4. The poor performance of KdV in modeling the phases suggests that a model with more accurate dispersion needs to be used. The role of dispersion is further discussed in each of the remaining sections.

Figure 4 contains plots comparing the KdV predictions with the experimental data for experiment #3. This experiment had an initial wave amplitude three times that of experiment #2 and therefore it is expected to demonstrate more nonlinear effects than experiment #2. The KdV predictions for this experiment are qualitatively similar to those in the previous experiment except that the amplitudes of the trailing waves are greatly overpredicted. This overprediction increases as the waves move away from the wave maker. These effects may be attributable to dissipative experimental effects that KdV cannot model. At each gauge, KdV accurately predicts the phases for the first couple of waves, but the error in phase prediction increases as time increases. The error in the KdV prediction for this experiment is an order of magnitude higher than the error in experiment #2.

3.2.2. The Serre Equations

Figure 5 contains a plot comparing results from numerical simulations of the Serre equations with the data for experiment #2. The Serre equations provide a much more accurate prediction for the experimental data than did KdV. However, just as with KdV, the Serre equations (a nondissipative system) overpredict the amplitudes of the trailing waves of elevation. This overprediction increases as the waves evolve down the tank and may be related to dissipative effects and the fact that the initial velocity was assumed to
Figure 3: Plots comparing results from a numerical solution of KdV (thin, black curves) with experimental data (thick, gray curves) corresponding to experiment #2. The vertical axes are scaled surface displacement, $\frac{2}{h_0} \eta$, and the horizontal axes are scaled time, $\sqrt{\frac{2}{h_0}} t - \frac{\xi}{h_0}$, with $\frac{\xi}{h_0} = 0$, 50, 100, 150, 200 for (a), (b), (c), (d), and (e) respectively.

Figure 4: Plots comparing results from a numerical solution of KdV (thin, black curves) with experimental data (thick, gray curves) corresponding to experiment #3. See the caption of Figure 3 for axes’ definitions.
be zero. The Serre equations accurately model the phase of the experimental data for the initial portions of the time series, but as time increases at each gauge, the phase error increases. However, the phase error is significantly less than in the KdV predictions. Additionally, the Serre equations predict a group behavior that is similar to what was observed in the experiments. Comparisons for experiment #3 are similar except that the amplitudes of the trailing waves are significantly overpredicted. Table 1 shows that the Serre equations provide more accurate predictions than does KdV. Surprisingly, the “fully nonlinear” Serre equations do not provide a significantly more accurate prediction than KdV for experiment #3 even though it is a more nonlinear experiment. This may be related to the fact that the unknown experimental initial velocities were set to zero (see Section 3.2.4 for a discussion of how velocities impact the model accuracy).

3.2.3. The Whitham Equation

Figure 6 demonstrates that the Whitham equation (including surface tension) does a very good job modeling the wave evolution in experiment #2. It accurately models both the initial wave of depression and the trailing dispersive waves. The phases are accurately reproduced for the entirety of each time series, including the group behavior of the trailing waves. This is likely due to the fact that the Whitham equation matches the unidirectional phase velocity of the Euler equations. Although the Whitham equation provides a very good prediction for the initial triangular wave and phases, it overestimates the amplitudes of most of the trailing dispersive waves.

The Whitham results for experiment #3 are included in Figure 7. This figure shows that the Whitham equation models the initial wave of depression and the phases of all trailing waves, but does not accurately model the amplitudes of the trailing waves of elevation. If the amplitude predictions were smaller, then the error values would be significantly smaller. Table 1 shows that the Whitham equation provides significantly more accurate predictions than the KdV and Serre equations for both experiments. Additionally, the table shows that including surface tension effects improves the Whitham prediction.

For experiment #2, the predictions obtained from the Sqrt Whitham equation are almost identical to those obtained from the Whitham equation. However there is a significant difference between the two predictions for experiment #3. This is caused by the fact that experiment #3 is more nonlinear than experiment #2. The Sqrt Whitham equation accounts for this more dramatically than does the Whitham equation because it is a more nonlinear model. Unfortunately, the Sqrt Whitham equation is less accurate than the Whitham equation for experiment #3. This establishes that the additional nonlinearity in the Sqrt Whitham equation is not the most appropriate form of nonlinearity for these experiments.

3.2.4. The Bidirectional Whitham Equations

The bidirectional Whitham models provide predictions that are similar to one another. Plots of results from these models are not included due to their similarity with the Whitham results shown above. Many observations can be made by examining the error results contained in Table 1. First, note that the
Figure 6: A plot of a numerical solution of the Whitham equation including surface tension (thin, black curves) and experimental data (thick, gray curves) corresponding to experiment #2. See the caption of Figure 3 for axes’ definitions.

Figure 7: A plot of a numerical solution of the Whitham equation including surface tension (thin, black curves) and experimental data (thick, gray curves) corresponding to experiment #3. See the caption of Figure 3 for axes’ definitions.
predictions obtained from the bidirectional models all have larger $E$ values than does the unidirectional Whitham equation for both experiments. This is attributed to the fact that the initial horizontal velocities from the experiments were not known and were assumed to be zero. The $E$ values change dramatically if nonzero values are used for the initial velocities. The errors can be decreased to less than 0.06 for experiment #2 and less than 0.9 for experiment #3 if the velocities are chosen to be proportional to the initial surface displacement. (The details of these results are not included since they were found by trial and error instead of in a scientific manner.) Recalling that the derivation of KdV establishes that horizontal velocities are proportional to surface displacement (at leading order, in the KdV regime) suggests that this choice improves the predictions is not a coincidence. This suggests that the bidirectional Whitham equations would outperform the Whitham equation if the initial horizontal velocities were known even though the experiment is essentially unidirectional.

Additional observations:

- All bidirectional Whitham systems very accurately model the experimental phases because they accurately reproduce the Euler phase velocity.
- All bidirectional Whitham systems accurately model the initial wave of depression.
- Including surface tension almost always improves the predictions.
- The ASMP and HP systems return very similar predictions. This is an interesting result given how little is known about the well posedness of the ASMP system. Though, perhaps it is not an unexpected result because the initial conditions are smooth, have relatively small amplitudes, and the models have the same asymptotic order.
- As with all of the other models, the bidirectional Whitham equations do not accurately model the amplitudes of the trailing waves. The amplitude overprediction may be related to the facts that all of the models are conservative and the experimental data appears to demonstrate dissipative effects. Many models of dissipation for small-amplitude, long waves have been developed, including [31, 32, 33, 34]. Although this is an interesting open question, these models and effects are outside the scope of this work.
- The higher-order MKD system provided predictions that are more accurate than the other bidirectional models and almost as accurate as the Whitham equation even when the initial horizontal velocities are assumed to be zero. By choosing the initial velocity to be proportional to the surface displacement, the MKD system provides the most accurate approximation to the experimental data. This suggests that nonlinearity of MKD form is necessary and that the leading-order nonlinearity utilized by the KdV and Whitham equations is not sufficient to accurately model these experiments.

4. Summary

We have compared laboratory measurements from two surface water-wave experiments on shallow water with predictions from the KdV, Serre, Whitham, and several generalized Whitham equations. We showed that predictions obtained from the Whitham equation (with or without surface tension) were significantly more accurate than those obtained from the KdV and Serre equations. The Whitham and generalized Whitham equations very accurately modeled the phases observed in the experiments. This is attributable to the fact that an accurate reproduction of the Euler phase velocity is necessary in order to accurately model these experiments. We showed that nonlinearity played an important role in these experiments. The predictions obtained from the linear theory provided the least accurate approximations to the experimental data. The nonlinearity in the KdV, Whitham, and Sqrt Whitham equations was insufficient to accurately model these experiments. However, the MKD system, which includes higher-order nonlinearity, can give the most accurate predictions. We showed that the inclusion of surface tension improved the predictions. Finally, we found that the bidirectional Whitham models could provide more accurate models if the initial horizontal velocities were chosen properly. The success of the bidirectional Whitham equations in modeling these experiments suggests that they are appropriate to model real world phenomena.
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