# Commutants mod Normed Ideals and $k_J(\tau)$

Dan-Virgil Voiculescu
UC Berkeley

D.-V. Voiculescu

supported in part by NSF grant DMS-1665534

I. the invariant $k_J(\tau)$

II. the commutant mod a normed ideal $\mathcal{E}(\tau; J)$

III. some uses of $k_J(\tau)$

IV. the hybrid generalization

# I.

## Normed ideals of compact operators

$(\mathcal{J}, |\ |_{\mathcal{J}})$

$(\mathcal{C}_p, |\ |_p)$ p-class $|T|_p = \left(\sum_j s_j^p\right)^{1/p}$

$(\mathcal{C}_p^-, |\ |_p^-)$ Lorentz $(p,1)$, $|T|_p^- = \sum_j s_j\, j^{-1+1/p}$

$|T|_\infty^- = \sum_j s_j\, j^{-1}$ Macaev norm

$s_1 \geq s_2 \geq \cdots$ eigenvalues of $(T^*T)^{1/2}$

$(\mathcal{J}^d, |\ |_{\mathcal{J}^d})$ dual, $\mathcal{J} \times \mathcal{J}^d \ni (X, Y) \longrightarrow \mathrm{Tr}\, XY$

# The modulus of quasicentral approximation

$\tau = (T_j)_{1 \le j \le n}$ bdd. operators, $\mathcal{J}$ normed ideal

$k_{\mathcal{J}}(\tau)$ = smallest $C \in [0,\infty]$ for which $\exists\, A_m \uparrow I$

$0 \le A_m \le I$, finite rank, $\max\limits_{1 \le j \le n} |[A_m, T_j]|_{\mathcal{J}} \xrightarrow[m \to \infty]{} C$

$\mathcal{J} = \mathcal{C}_p$ , $k_p(\tau)$ , $\mathcal{J} = \mathcal{C}_p^-$ , $k_p^-(\tau)$

$p = \infty$ $k_\infty^-(\tau)$ Macaev norm case

## General properties

1° $\quad p \longrightarrow k_p^-(\tau)$

decreasing function of $p \in [1, \infty]$

2°. there is $p_0 \in [1, \infty]$ so that

$p \in [1, p_0) \Longrightarrow k_p^-(\tau) = \infty$

$p \in (p_0, \infty] \Longrightarrow k_p^-(\tau) = 0$

3°. $\tau - \tau' \in \mathcal{J} \Longrightarrow k_{\mathcal{J}}(\tau) = k_{\mathcal{J}}(\tau')$

assuming finite rank ops. dense in $\mathcal{J}$.

4°. $\tau = \tau^*$ and finite rank dense in $\mathcal{J}$, then:

$$k_{\mathcal{J}}(\tau) > 0$$

$$\Updownarrow$$

$$\exists\ Y_j = Y_j^* \in \mathcal{J}^d, \quad 1 \le j \le n$$

$$\operatorname{Tr}\ i \underbrace{\sum_j [T_j, Y_j]}_{\mathcal{C}_1 \overset{\Uparrow}{+} \mathcal{B}(\mathcal{H})_+} > 0$$

5°. $p > 1 \Longrightarrow k_p(\tau) \in \{0, \infty\}$.

6°. $k_\infty^-(\tau) \le 2 \|\tau\| \log(2n+1)$

7°. $\mathcal{C}_\infty^- \subsetneqq \mathcal{J} \Longrightarrow k_{\mathcal{J}}(\tau) = 0$

8°. $\tau$ $n$-tuple of commuting hermitian ops.

$$(k_n^-(\tau))^n = \gamma_n \int_{\mathbb{R}^n} m(s)\, d\lambda(s)$$

$\lambda$: $n$-dim Lebesgue measure

$0 < \gamma_n < \infty$

$m(s)$: multiplicity of Lebesgue absolutely continuous part

9°. $\tau$ $n$-tuple of commuting hermitian ops.

$\mathcal{C}_n^- \subsetneqq \mathcal{J} \Longrightarrow k_{\mathcal{J}}(\tau) = 0$

10°. $S_1, \dots, S_n$ Cuntz isometries, $n \geq 2$.

$k_\infty^-(S_1, \dots, S_n) > 0$.

# II.

## $\mathcal{E}(\tau;\mathcal{J})$ the commutant of $\tau$ mod $\mathcal{J}$

$$\mathcal{E}(\tau;\mathcal{J}) = \{X \in \mathcal{B}(\mathcal{H}) \mid [X,T_j] \in \mathcal{J}, 1 \le j \le n\}$$

$\tau = \tau^*$

$$|||X||| = \|X\| + \max_{1\le j\le n} |[X,T_j]|_{\mathcal{J}}.$$

Banach $*$-algebra (not $C^*$ in general).

$\mathcal{K}(\tau;\mathcal{J}) = \mathcal{E}(\tau;\mathcal{J}) \cap \mathcal{K}$ the compact ideal of $\mathcal{E}(\tau;\mathcal{J})$

$\mathcal{E}/\mathcal{K}(\tau;\mathcal{J}) = \mathcal{E}(\tau;\mathcal{J})/\mathcal{K}(\tau;\mathcal{J})$ the Calkin algebra of $\mathcal{E}(\tau;\mathcal{J})$

## $k_J(\tau)$ and approximate units of $\mathcal{K}(\tau; J)$

$k_J(\tau) = 0$

$\Updownarrow$

$\mathcal{K}(\tau; J)$ has approximate unit of norm one

$A_m \in \mathcal{K}(\tau; J)$, $|||A_m||| \leq 1$, $m \in \mathbb{N}$

$|||A_m K - K||| \to 0$, $|||K A_m - K||| \to 0$, $\forall K \in \mathcal{K}(\tau; J)$

$k_J(\tau) < \infty$

$\Updownarrow$

$\mathcal{K}(\tau; J)$ has bounded approximate unit

$A_m \in \mathcal{K}(\tau; J)$, $\sup_m |||A_m||| < \infty$

$|||A_m K - K||| \to 0$, $|||K A_m - K||| \to 0$, $\forall K \in \mathcal{K}(\tau; J)$

Finite rank ops. dense in $J$ assumption.

# $\mathfrak{E}/\mathcal{K}(\tau;\mathfrak{J})$

many similarities between

| | | |
|---|---|---|
| $\mathcal{K}$ | $\mathcal{B}$ | $\mathcal{B}/\mathcal{K}$ |
| $\mathcal{K}(\tau;\mathfrak{J})$ | $\mathfrak{E}(\tau;\mathfrak{J})$ | $\mathfrak{E}/\mathcal{K}(\tau;\mathfrak{J})$ |

notation: $\mathcal{R}$ finite rank ops.

$$p : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{K}$$

$\mathcal{B}/\mathcal{K} \supset p(\mathfrak{E}(\tau;\mathfrak{J})) \simeq \mathfrak{E}/\mathcal{K}(\tau;\mathfrak{J})$ algebraic iso.

$\mathfrak{J}^{(0)}$ closure of $\mathcal{R}$ in $\mathfrak{J}$

$$\tilde{\mathfrak{J}} = \{ X \in \mathcal{K} \mid \sup_{P \in \mathcal{P} \cap \mathcal{R}} |PX|_{\mathfrak{J}} < \infty \}$$

$\mathcal{P}$ hermitian projections

A). <u>Corona</u> assuming $k_J(\tau) < \infty$

$\mathcal{K}(\tau; J^{(0)})$ is a closed ideal in $\mathcal{E}(\tau; \tilde{J})$.

$\mathcal{E}(\tau; \tilde{J})$ is double centralizer of $\mathcal{K}(\tau; J^{(0)})$.

If $J^{(0)} = \tilde{J}$, $\mathcal{E}/\mathcal{K}(\tau; J)$ corona of $\mathcal{K}(\tau; J)$.

B). <u>$C^*$-algebras</u> assuming $J = J^{(0)}$

a) if $k_J(\tau) = 0$ then $\mathcal{E}/\mathcal{K}(\tau; J)$ and $p(\mathcal{E}(\tau; J))$ are $C^*$-algebras and are isometrically isomorphic

b) if $k_J(\tau) < \infty$ then $p(\mathcal{E}(\tau; J))$ is a $C^*$-algebra and is isomorphic to $\mathcal{E}/\mathcal{K}(\tau; J)$ (not isometrically)

<u>Cor.</u> $\mathcal{E}/\mathcal{K}(\tau; \mathcal{C}_\infty^-)$ isomorphic to $C^*$-algebra (all $\tau$ !)

C). Assuming $\mathcal{J} = \mathcal{J}^{(0)}$ and $k_{\mathcal{J}}(\tau) = 0$ then $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$ is countably degree-1 saturated (Farah-Hart property).

D). Duality

a) assume $\mathcal{R}$ dense in $\mathcal{J}$ and $\mathcal{J}^d$ and $k_{\mathcal{J}}(\tau) = 0$ then $\mathcal{E}(\tau;\mathcal{J}) \simeq$ bidual of $\mathcal{K}(\tau;\mathcal{J})$

b) assume $\mathcal{J}$ reflexive and $k_{\mathcal{J}}(\tau) = 0$ then $\mathcal{E}(\tau;\mathcal{J})$ has unique predual.

# $K_0(\mathcal{E}(\tau; \mathfrak{I}))$ simple examples

$\tau_n$ $n$-tuple of multiplication operators by the coordinate functions in $L^2([0,1]^n, d\lambda_n)$.

$K_0(\mathcal{E}(\tau_n; \mathcal{K})) = K_0(\mathcal{E}/\mathcal{K}(\tau_n; \mathcal{K})) = 0$ since $\mathcal{E}/\mathcal{K}(\tau_n; \mathcal{K})$ is the Paschke dual of $C([0,1]^n)$ and $[0,1]^n$ is contractable. When $K_0(\mathcal{E}/\mathcal{K}(\tau_n; \mathfrak{I})) \neq 0$ we infer $\mathcal{E}/\mathcal{K}(\tau_n; \mathfrak{I})$ is not a smooth subalgebra of the Paschke dual $\mathcal{E}/\mathcal{K}(\tau_n; \mathcal{K})$.

$\mathfrak{F}_n$ ordered group of Lebesgue measurable $f : [0,1]^n \to \mathbb{Z}$ in $L^\infty([0,1]^n, d\lambda_n)$ with a.e. equivalence

$$\mathfrak{F}_n \simeq K_0((\tau_n)')\,.$$

1°. $n=1$, $\mathcal{J}=\mathcal{C}_1$

$$K_0(\mathcal{E}(\tau_1;\mathcal{C}_1)) \simeq K_0(\mathcal{E}/\mathcal{K}(\tau_1;\mathcal{C}_1)) \simeq \mathcal{F}_1$$

$P$ projection in $\mathfrak{M}_n(\mathcal{E}(\tau_1,\mathcal{C}_1))$

$[P]_0 \rightsquigarrow$ multiplicity of Lebesgue abs. cont. part of $P(T_1 \otimes I_m)P$

Kato-Rosenblum thm. corollary used.

2°. $n=1$, $\mathcal{J} \neq \mathcal{C}_1$ (i.e. $\mathcal{C}_1 \subsetneqq \mathcal{J}$)

$$K_0(\mathcal{E}(\tau_1,\mathcal{J})) = K_0(\mathcal{E}/\mathcal{K}(\tau_1,\mathcal{J})) = 0.$$

3° $n \geq 3$, $\mathfrak{J} = \mathcal{C}_n^-$

$$K_0(\mathcal{E}(\tau_n; \mathcal{C}_n^-)) = K_0(\mathcal{E}/\mathcal{K}(\tau_n; \mathcal{C}_n^-)) \simeq \mathcal{F}_n \oplus \underset{\text{unknown}}{\mathcal{X}_n}$$

uses our perturbation results based on $k_n^-$.

4° $n = 2$, $\mathfrak{J} = \mathcal{C}_2$

$$K_0(\mathcal{E}(\tau_2; \mathcal{C}_2)) \longrightarrow L^1_{real}([0,1]^2, d\lambda_2)$$

$$[P]_0 \rightsquigarrow g_{P(T_1+iT_2)P}$$ Pincus principal function

nontrivial homomorphism with $\infty$ rank range

$X = P(T_1 + iT_2)P$ almost normal operator

self-commutator $[X^*, X] \in \mathcal{C}_1$.

# III.

## Some uses of $k_J(\tau)$

1°. $\tau, \tau'$ $n$-tuples of commuting hermitian ops.

$\tau - \tau' \in \mathcal{C}_n^- \Longrightarrow (\tau)_{ac}$ unitarily equiv. $(\tau')_{ac}$

Lebesgue abs. cont. parts

$n=1$ corollary of Kato–Rosenblum thm.

$n>1$ generalization uses $k_n^-$ technique.

2°. $\tau$ $n$-tuple of commuting hermitian ops.

$J \supsetneqq \mathcal{C}_n^- \Longrightarrow \delta$ diagonal $n$-tuple $\quad \delta - \tau \in J$

$n=1$ Kuroda–Weyl–v. Neumann thm.

$n>1$ generalization uses $k_J$ technique

3°. Connes' technical use of $k_n^-(\tau)$ and of results on $\mathcal{C}_n^-$-perturbations of $n$-tuples of commuting hermitian ops. in the spectral characterization of manifolds

4°. further perturbation results, including analogues of wave-operators using $k_J$ some even for non-integer dimension.

# IV.

## The hybrid generalization

$\mathcal{J}$ normed ideal $\rightsquigarrow (\mathcal{J}_1, \dots, \mathcal{J}_n)$ n-tuple of normed ideals

$k_{(\mathcal{J}_1,\dots,\mathcal{J}_n)}(\tau)$ = smallest $C \in [0,\infty]$ for which

$\exists\, A_m \uparrow I$, $0 \le A_m \le I$, $A_m \in \mathcal{R}$ so that

$$\max_{1\le j\le n} |[A_m, T_j]|_{\mathcal{J}_j} \xrightarrow[m\to\infty]{} C$$

$k_{p_1\dots p_n}(\tau)$, $k^-_{p_1\dots p_n}(\tau)$

$\tau = (T_j)_{1\le j\le n}$ , $\tau' = (T_j')_{1\le j\le n}$

n-tuples of commuting hermitian ops.

a) $T_j - T_j' \in \mathcal{C}^-_{p_j}$ , $p_1^{-1} + \dots + p_n^{-1} = 1$

$\Longrightarrow \quad \tau_{ac} \underset{\text{equiv}}{\overset{\text{unitary}}{\sim}} \tau'_{ac}$

b) $\left(k^-_{p_1,\dots,p_n}(\tau)\right)^n = \gamma_{p_1\dots p_n} \int_{\mathbb{R}^n} m(s)\, d\lambda(s)$

$0 < \gamma_{p_1\dots p_n} < \infty$ if $p_1^{-1} + \dots + p_n^{-1} = 1$

Singular integrals for proof of

$\tau_{p_1,\dots,p_n} > 0$ if $p_1^{-1} + \dots + p_n^{-1} = 1$

$Y_j$ operator in $L^2([-1,1]^n, d\lambda)$ defined by kernel

$$K_j(x,y) = \operatorname{sign}(x_j - y_j)\,|x_j - y_j|^{p_j - 1}\Big(\sum_{1 \le k \le n} |x_k - y_k|^{p_k}\Big)^{-1}$$

$$Y_j \in \left(\mathcal{C}^-_{p_j}\right)^{dual} \qquad 1 \le j \le n$$

# References

1°. V. Perturbations of operators, connections w. singular
integrals, hyperbolicity and entropy.
in Harmonic Analysis and Discrete Potential Th.
Plenum Press 1992, p. 181–191

survey of older work contains
many references

2°. V. K-theory and perturbations of absolutely continuous spectra, arXiv: 1606.00520

3°. V. Almost normal operators mod Hilbert-Schmidt and the K-theory of the Banach algebras $E\Lambda(\Omega)$, J. Noncommut. Geom. 8 (2014) no. 4, 1123–1145, arXiv: 1112.4930

4°. V. Lebesgue decomposition of functionals and unique preduals for commutants modulo normed ideals, Houston J. Math. 43 (2017), no. 4 1251–1262 arXiv: 1608.07228

(Ref-3

5° Connes, On the spectral characterization of manifolds, J. Noncommut. Geom. 7 (2013) no. 1, 1 – 82

6°. V. Countable degree-1 saturation of certain $C^*$-algebras which are coronas of Banach algebras.
Groups Geom. Dyn. 8 (2014) no. 3
985 – 1006
arXiv: 1310.4862

(Ref-4)

7°. V. Hybrid normed ideal perturbations of n-tuples of operators I. J. Geom. Phys. 128(2018), 169–184

8°. V. Hybrid normed ideal perturbations of n-tuples of operators II: weak wave operators. arXiv: 1801.00490