# Boolean extreme values

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Joint work with Dan-Virgil Voiculescu

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# Classical extreme value theory (Motivation)

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- In general terms EVT studies "extreme" occurrences in an stochastic process.
- **Example**: Given a sequence of i.i.d. $X_1, X_2, \ldots,$ consider $M_n = \bigvee_{i=1}^n X_i$. Is there a sequence of normalization constants $a_n, b_n$ such that

$$\frac{M_n - b_n}{a_n}$$

has a limiting distribution?

# Max-convolution

We must know how to compute the distribution of the supremum of a set of independent random variables.

- (**Max-convolution**) Let $X, Y$ be independent. Since $P(X \vee Y \leq t) = P(X \leq t, Y \leq t)$

$$F_{X \vee Y}(t) = F_X(t) F_Y(t).$$

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$$F_{X \vee Y}(t) = F_X(t) F_Y(t).$$

- (**Local**) To know the value of $F_{X \vee Y}$ at $t$, it is enough to know $F_X$ and $F_Y$ at $t$.

- Hence, the **semigroup** $([0,1], \cdot)$ encodes the information of the max-convolution in the classical case.

# Solution to the problem

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- Fréchet's distribution is the only **positively supported distribution**; its distribution function, of parameter $\alpha$, is defined as follows

$$\Phi_\alpha(x) = \begin{cases} 0 & x < 0, \\ \exp(-x^{-\alpha}) & x \geq 0. \end{cases}$$

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- The domains of attraction for each limiting distribution were nicely characterized.

# Non-commutative analogue

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# Non-commutative analogue

- Within probability theory there are extreme quantities that are of interest in the non-commutative context. E.g. The maximum eigenvalue of a random matrix.
- For us, random variables are operators over Hilbert spaces and once we fix a state, their distribution is determined.
- Given two non-commutative random variables, how do we construct their supremum? (With respect to which order do we take it?)

# Non-commutative analogue

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- (S. Sherman, 1951) If the s.a. operators in a C*-algebra form a lattice, then the C*-algebra is abelian.
- (P. Olson, 1971) The self adjoint operators of a von Neumann algebra form a conditionally complete lattice with respect to the **spectral order**.

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- So we have that

$$E(X \vee Y;(-\infty,t]) = E(X;(-\infty,t]) \wedge E(Y;(-\infty,t)).$$

# Free extremes

G. Ben Arous, D.V. Voiculescu. *Free Extreme Values*, Ann. Probab, Vol. 34, No. 5, 2006:

## Definition

If $F(t)$ and $G(t)$ then their free max-convolution is given by

$$H(t) = \max(0, F(t) + G(t) - 1).$$

# Free extremes

**Theorem (G. Ben Arous, D. V. Voiculescu, 2006)**

Any free max-stable law is of the same type of one of the following:

- Exponential: $F(x) = (1 - e^{-x})_+$
- The Pareto distribution: $F(x) = (1 - x^{-\alpha})_+$ for some $\alpha > 0$.
- The Beta law $F(x) = 1 - |x|^{\alpha}$ for $-1 \leq x \leq 0$ and some $\alpha > 0$.

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# Boolean independence

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- Rule for computing mixed moments : Let $(\mathcal{A}_i)_{i\in I}$ be subalgebras of a $*$-probability space $(\mathcal{A}, \phi)$. This subalgebras are Boolean independent if

$$\phi(X_1 \cdots X_n) = \phi(X_1) \cdots \phi(X_n),$$

whenever $X_k \in A_{i(k)}$ and $i(k) \neq i(k+1)$ for $k = 1, \ldots, n-1$.

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  whenever $X_k \in A_{i(k)}$ and $i(k) \neq i(k+1)$ for $k = 1, \ldots, n-1$.
- The above condition enforces to consider **non-unital** algebras. For if $1 \in \mathcal{A}$ we would have

$$\phi(X^2) = \phi(X1X) = \phi(X)^2\phi(1).$$

# Boolean independence

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  whenever $X_k\in A_{i(k)}$ and $i(k)\neq i(k+1)$ for $k=1,\ldots,n-1$.
- We must also consider **non-tracial** states. For if $\phi$ is tracial we would have

$$\phi(X^2)\phi(Y)=\phi(X^2Y)=\phi(XYX)=\phi(X)^2\phi(Y).$$

# The Boolean product

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- If $p_i$ is the rank-1 projection in $\mathcal{H}_i$ on $\langle \xi_i \rangle$, we consider the inclusions

$$(\mathcal{B}(\mathcal{H}_1), \xi_1) \hookrightarrow (\mathcal{B}(\mathcal{H}), \xi) \hookleftarrow (\mathcal{B}(\mathcal{H}_2, \xi_2))$$

given by $x \mapsto x \otimes p_2$, or $x \mapsto p_1 \otimes x$, depending on which $\mathcal{A}_i$ is $x$ in.

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- The images of this inclusions are Boolean independent.

# Boolean convolution

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- In the Boolean world, the role of the $R$-transform is substituted by the self-energy transform

$$\mu \mapsto K_\mu(z) = z - \frac{1}{G_\mu(z)},$$

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where $G_\mu(z)$ is the Cauchy transform of $\mu$.

- If $\mu$ and $\nu$ are compactly supported then

$$K_{\mu \uplus \nu}(z) = K_\mu(z) + K_\nu(z).$$

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# Method

Let $X$ and $Y$ be random variables independent in some sense (tensor, free, Boolean, monotone).

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What's left to do? (A lot!)

- One has to understand what is going on at the level of operators.
- Once the max-convolution is defined, one has to get a strong grasp on it to find the max-stable laws, domains of attraction, etc.

# Technical considerations in the Boolean case

In the inclusion determined by the Boolean product

$$\mathcal{B}(\mathcal{H}_1) \ni X \longmapsto \tilde{X} \in \mathcal{B}(\mathring{\mathcal{H}}_1 \oplus \xi \oplus \mathring{\mathcal{H}}_2) = \mathcal{B}(\mathcal{H})$$

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the **kernel** of $X$ **gets enlarged**, so if $V : \mathcal{H}_1 \to \mathcal{H}$ is the isometric inclusion we have that $\tilde{X} = VXV^*$ and

- If $t < 0$ then

$$E(\tilde{X}, (-\infty, t]) = VE(X; (-\infty, t])V^*.$$

- While if $t \geq 0$ then

$$E(\tilde{X}; (-\infty, t]) = VE(X; (-\infty, t])V^* + P_{\mathring{\mathcal{H}}_2}.$$

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We restrict our study to random variables with distributions **supported** in $\mathbb{R}_{\geq}0$.

# Technical considerations in the Boolean case

**Recall the objective**: Given a sequence of i.i.d. $X_1, X_2, \ldots,$ consider $M_n = \bigvee_{i=1}^n X_i$. Is there a sequence of normalization constants $a_n, b_n$ such that

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has a limiting distribution?

Boolean probability is a **non-unital** theory. So **no shifts** will be considered, i.e. $b_n = 0$ for all $n$.

# Boolean max-convolution

The Boolean max-convolution also turns out to be “local” for distribution functions.

**Definition**

If $F_1, F_2$ are distribution functions on $[0, \infty)$ then their Boolean max-convolution is defined by

$$(F_1 \cup\!\!\!\!\vee\, F_2)(t) = F_1(t) \,\triangle\!\!\!\!\vee\, F_2(t)$$

where $\triangle\!\!\!\!\vee$: $(p, q) \mapsto \frac{1}{p^{-1}+q^{-1}-1}$.

# Boolean max-convolution

## Definition

Let $\Delta_+$ be the set of distribution functions supported on $\mathbb{R}_{\geq 0}$. A distribution function $F \in \Delta_+$ is a Boolean max-stable distribution function if for some $G \in \Delta_+$ there are constants $a_n > 0$, $n \in \mathbb{N}$ so that

$$\underbrace{(G \uplus \cdots \uplus G}_{n})(a_n t) \to F(t),$$

for all $t \geq 0$.

# Results (Our transform)

## Lemma

The semigroups $([0,1], \uplus)$ and $([0,1], \cdot)$ are isomorphic. The map $\chi : [0,1] \to [0,1]$ given by

$$\chi(x) = \exp(1 - x^{-1})$$

is an isomorphism which is also an order preserving homeomorphism. The inverse isomorphism, which is also order-preserving is given by

$$\chi^{-1}(y) = (1 - \log(y))^{-1}.$$

# Results (Transfer from classical probability)

## Observation

The map

$$X(F) = \begin{cases} \chi(F(x)) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

preserves $\Delta_+$, and is an isomorphism between $(\Delta_+, \uplus)$ and $(\Delta_+, \cdot)$.

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Recall that, from the max-stable distributions in classical probability, Fréchet's distribution is the only **positively supported distribution**; its distribution function, of parameter $\alpha$, is defined as follows

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# Results (Max-stable laws)

**Theorem (JGV, Voiculescu, 2017)**

$F \in \Delta_+$ is a Boolean max-stable law if and only if

$$F(t) = 1 - \frac{\lambda}{t^\alpha + \lambda}$$

where $\lambda > 0$ and $\alpha > 0$. This distributions are called Dagum distributions (or log-logistic distributions) and have been widely studied in the literature of classical probability.

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Since in the Boolean CLT the limiting distributions turn out to be of the form $\frac{1}{2}(\delta_{-1} + \delta_1)$, it is somehow surprising that in this case the stable laws have heavy tails.

# Results (Domains of attraction)

## Theorem (Gnedenko, 1943)

*$F \in Dom(\Phi_\alpha)$ if and only if $1 - F \in RV_{-\alpha}$. In this case*

$$F^n(a_n x) \to \Phi_\alpha(x),$$

*with*

$$a_n = (1/(1-F))^{\leftarrow}(n).$$

*Where $RV_a$ denotes the set of regularly varying functions of index $a$, i.e., the set of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that, for every $x > 0$ it holds that*

$$x^a = \lim_{t\to\infty} \frac{f(tx)}{f(t)}.$$

# Results (Domains of attraction)

**Theorem (JGV, Voiculescu 2017)**

*$G \in \Delta_+$ is in the Boolean domain of attraction of the Dagum distribution function $F(t) = 1 - \frac{1}{1+t^\alpha}$ if and only $1 - F$ is regularly varying of index $-\alpha$.*

# Index

# Within the Boolean World

- (Finite dimensional approximation) F. Benaych-Georges, T. Cabanal-Duvillard, *A matrix interpolation between classical and free max operations. I. The univariate case*, J. Theoretical Probab., Vol. 23, No. 2, 2010.
- (Order statistics) G. Ben Arous, V. Kargin. *Free point processes and free extreme values*, Probab. and Rel. Fields, Vol. 147, No. 1-2, 2010.
- (Insight into the stable-laws) J. Grela, M. A. Nowak. *On relations between extreme value statistics, extreme random matrices and Peak-Over-Threshold method*, arXiv: 1711.03459, 2017.

# Other frameworks

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  - (Finite-free probability) If $p_n(x)$ and $q_n(x)$ are polynomials of degree $n$. If $A$ and $B$ are real symmetric $n \times n$ matrices with $p_n(x) = \chi_A(x)$ and $q_n(x) = \chi_B(x)$ then

$$p_n(x) \boxplus_n q_n(x) = \mathbb{E}_U[xI - A - UBU^*].$$

# Other frameworks

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$$p_n(x) \boxplus_n q_n(x) = \mathbb{E}_U[xI - A - UBU^*].$$

    - Can we make sense of extreme value theory in this context? The main obstruction is that this theory is currently only at the level of convolutions of polynomials (and measures), but has no variables.

# Other frameworks

- (Tropical context) A. Rosenmann, F. Lehner, A. Peperko. *Polynomial convolutions in max-plus algebra*, arXiv:1802.07373, 2018.
  - Define $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and consider $(\mathbb{R}_{\max}, \oplus, \odot)$, where $a \oplus b = \max\{a, b\}$ and $a \oplus b = a + b$. This is a semi-ring.

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  - Max-polynomials are of the form

$$p(x) = \bigoplus_{k=0}^{d} a_k \odot x^k, \quad a_k \in \mathbb{R}.$$

# Other frameworks

- (Tropical context) A. Rosenmann, F. Lehner, A. Peperko. *Polynomial convolutions in max-plus algebra*, arXiv:1802.07373, 2018.
  - Define $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and consider $(\mathbb{R}_{\max}, \oplus, \odot)$, where $a \oplus b = \max\{a, b\}$ and $a \oplus b = a + b$. This is a semi-ring.
  - Max-polynomials are of the form

$$p(x) = \bigoplus_{k=0}^{d} a_k \odot x^k, \quad a_k \in \mathbb{R}.$$

  - An analogous convolution can be defined between these polynomials. In this case, the induced convolution in distributions coincides with the free max-convolution.

Thank you!

J. Garza Vargas, D. V. Voiculescu. *Boolean extremes and Dagum distributions*, arXiv:1711.06227, 2017.