Spectral Triples, Quantum Compact Metric Spaces, and the Sierpinski Gasket

Therese Landry
Faculty Advisor: Dr. Michel Lapidus

University of California, Riverside

tland004@ucr.edu

WCOAS 2018, October 6, 2018

Overview

Finite Approximations of Fractals

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Noncommutative Geometry

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- Spectral Triples

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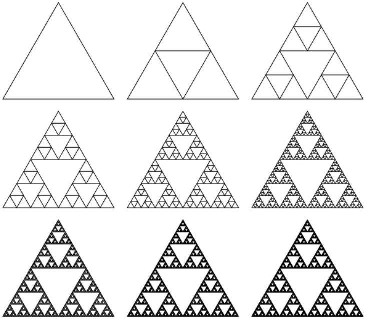
Sierpinski Gasket

Definition



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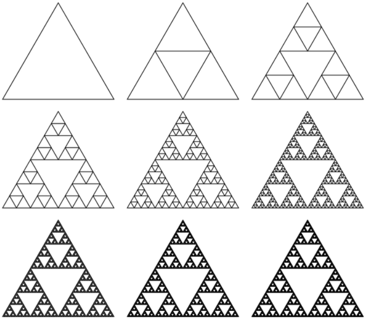

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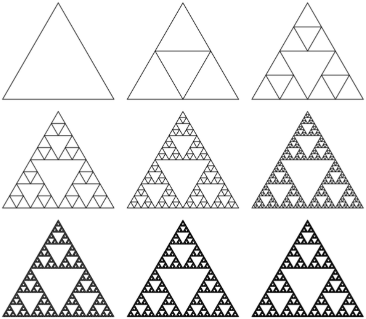

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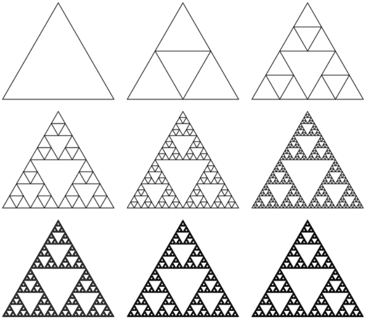

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The *Sierpinski gasket* K is the unique nonempty compact subset of $\mathbb{R}^2$ such that $K = \cup_{i=1}^{3} F_i(K)$.

The space of continuous complex-valued functions on a compact Hausdorff space is a C*-algebra.

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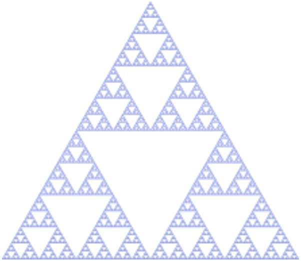

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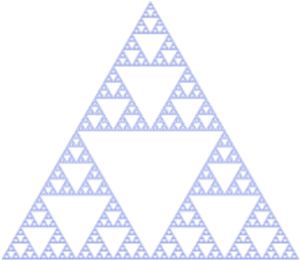

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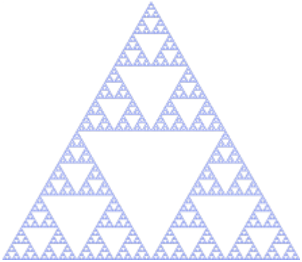

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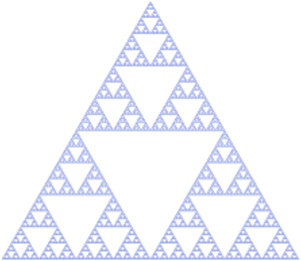

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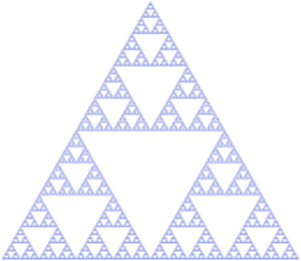

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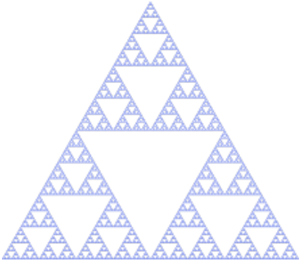

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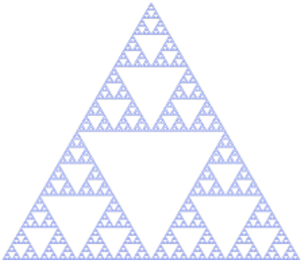

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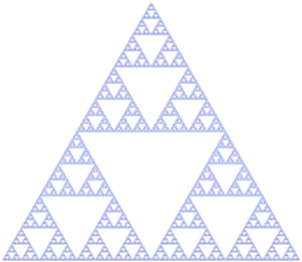

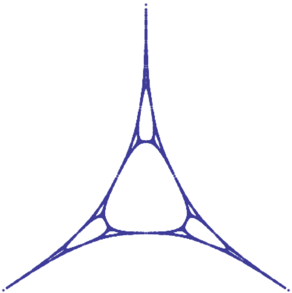

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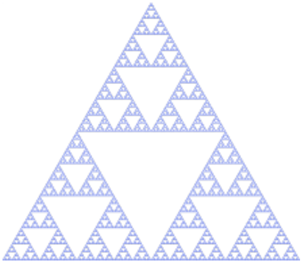

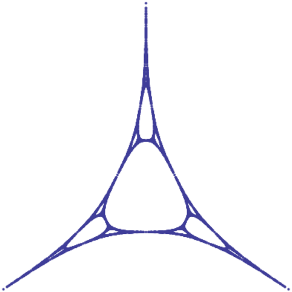

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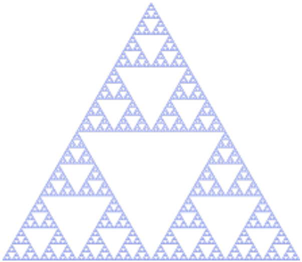

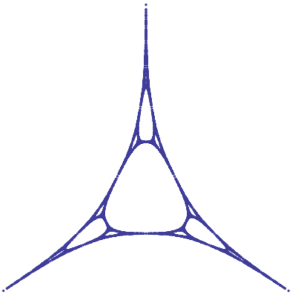

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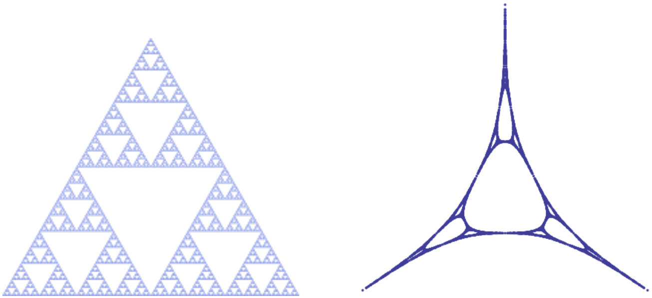

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The Sierpinski gasket and the harmonic Sierpinski gasket both satisfy these conditions.

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A *quantum compact metric space* $(\mathcal{A}, L)$ is a pairing of a unital C*-algebra with a Lip-norm L such that

$$\{a \in \mathfrak{sa}(\mathcal{A}) : L(a) = 0\} = \mathbb{R}1_{\mathcal{A}}$$

and mk_L metrizes the weak* topology of $\mathcal{S}(\mathcal{A})$.

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Standard ultrametrics on the Cantor set can be recovered from a noncommutative, or quantum, metric.

Definition

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An *Approximately Finite*, or *AF*, *Algebra* is an inductive limit of finite dimensional C*-algebras.

Definition

A *conditional expectation* $\mathbb{E}(\cdot|\mathcal{B}) : \mathcal{A} \to \mathcal{B}$ onto $\mathcal{B}$, where $\mathcal{A}$ is a C*-algebra and $\mathcal{B}$ is a C*-subalgebra of $\mathcal{A}$, is a linear positive map of norm 1 such that for all $b, c \in \mathcal{B}$ and $a \in \mathcal{A}$ we have

$$\mathbb{E}(bac|\mathcal{B}) = b\mathbb{E}(a|\mathcal{B})c.$$

Theorem (Aguilar, Latrémolière 2016)

Let $\mathcal{A}$ be a unital C^*-algebra which is the closure of $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$, where for all n, $\mathcal{A}_n$ is a finite-dimensional unital C^*-subalgebra of $\mathcal{A}$.

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$$\mathcal{C} = \{(z_n)_{n \in \mathbb{N}} : z_n \in \mathbb{Z}_2\} = \mathbb{Z}_2^{\mathbb{N}}$$

with the product topology.

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- η_n is a projection
- $u_n := 2\eta_n - 1_{C(\mathcal{C})}$ is a self-adjoint unitary in $C(\mathcal{C})$

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- $C(\mathcal{C})$ is the closure of $\cup_{n\in\mathbb{N}}\mathcal{A}_n$

A faithful tracial state can be defined on $C(\mathcal{C})$.

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- $\mathcal{C}$ is the union of $2^{\#F}$ disjoint translates of F
- with respect to addition modulo 1, the set $\mathcal{C} = \Pi_{n \in \mathbb{N}} \mathbb{Z}^2$ is a group,

For every finite, nonempty $F \subset \mathbb{N}$, let $\#F$ denote the cardinality of F and

$$\lambda\left(\Pi_{j\in F}\eta_j\right) := 2^{-\#F}.$$

Observe that

- $\Pi_{j\in F}\eta_j$ is the indicator function of the subset

$$\{(z_n)_{n\in\mathbb{N}} \in \mathcal{C} : z_j = 1 \quad \forall j \in F\}$$

- $\mathcal{C}$ is the union of $2^{\#F}$ disjoint translates of F
- with respect to addition modulo 1, the set $\mathcal{C} = \Pi_{n\in\mathbb{N}}\mathbb{Z}^2$ is a group, hence $\mathcal{C}$ compact implies there exists a unique Haar probability measure which defines by integration a faithful tracial state λ on $C(\mathcal{C})$

Lemma (Aguilar, Latrémolière 2015)

If $C(\mathcal{C})$ is endowed with the inner product

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then $u_n \in \mathcal{A}_n^{\perp}$ for all $n \in \mathbb{N}$. Moreover, $\left(\prod_{j\in F} u_j\right)_{F\in\mathcal{F}}$, where $\mathcal{F}$ is the set of nonempty finite subsets of $\mathbb{N}$, is an orthonormal family of $L^2(C(\mathcal{C}), \lambda)$.

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$$\mathrm{mk}_{L^{\beta}_{(\mathcal{A}_n,\alpha_n)_{n\in\mathbb{N}},\lambda}}(x,y) = \begin{cases} 0 \quad \textit{if} \quad x = y, \\ 2\beta(\min\{n \in \mathbb{N} : x_n \neq y_n\}) \quad \textit{otherwise}. \end{cases}$$

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By construction, $\mathsf{mk}_{L^{\beta}_{\mathcal{T},\lambda}}$ is an ultrametric on $\mathcal{C}$.

Definition

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There exists a generalization of the Gromov-Hausdorff distance to the quantum compact metric space, due to Latrémolière and built on the work of Rieffel, which is a complete metric up to full quantum isometry that induces the same topology as the Gromov-Hausdorff distance in the classical setting.

Theorem (Latrémolière 2016)

If $(\mathcal{A}, L_{\mathcal{A}})$ *and* $(\mathcal{B}, L_{\mathcal{B}})$ *are quantum compact metric spaces with Gromov-Hausdorff propinquity* $\Lambda((\mathcal{A}, L_{\mathcal{A}}), (\mathcal{B}, L_{\mathcal{B}})) = 0$*,*

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Theorem (Aguilar, Latrémolière 2016)

Let $\mathcal{A}$ be a unital C^*-algebra which is the closure of $\bigcup \mathcal{A}_n$, where each $\mathcal{A}_n$ is a finite-dimensional C^*-subalgebra of $\mathcal{A}$.

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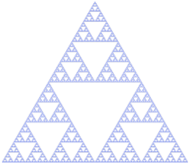
Directions for Future Work



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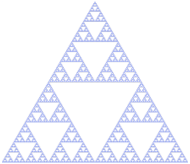

A unital commutative C*-algebra is an AF algebra

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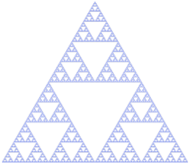

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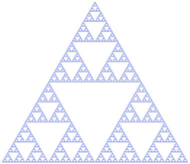

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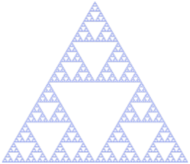

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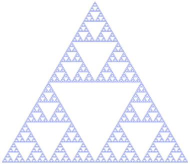

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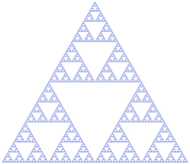
Can the Gromov-Hausdorff propinquity be used to write $C(K)$ *as a limit of* C^**-algebras? what kinds of* $C*-$ *algebras? If* $C(K)$ *is not an AF algebra, what kind of* C^**-algebra is it?*

Directions for Future Work



Let $\oplus X_j$ be a decomposition of the Sierpinski Gasket into parametrized curves, $(C(X_j), \mathcal{H}_j, D_j)$ a spectral triple for parametrized curves on X_j, and $(C(X), \oplus_j \mathcal{H}, \oplus_j D_j)$ with representation $\oplus_j \pi_j$ the spectral triple for the Sierpinski Gasket constructed by Antonescu, Christiansen, and Lapidus.

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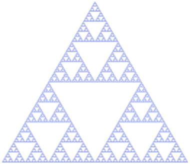

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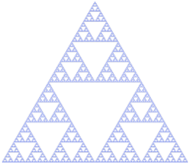
Under what conditions can $\oplus_j D_j$ be used to construct a Lip-norm which equips $C(K)$ with a quantum compact metric space structure?

Directions for Future Work



In the setting of the Sierpinski gasket, what aspects of geometry can be recovered via the Gromov-Hausdorff propinquity?

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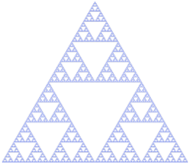

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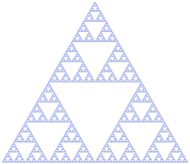

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How does the information obtained from spectral triples for the Sierpinski gasket compare with that gained by viewing it as a quantum compact metric space? Can new bridges between these two noncommutative notions be built in the context of the Sierpinski gasket?

Directions for Future Work

What other fractals can be finitely approximated by noncommutative means?

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The *height* $\varsigma(\gamma|L_{\mathcal{A}}, L_{\mathcal{B}})$ of γ is given by

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Fix $\theta \in (0,1) \setminus \mathbb{Q}$.

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$$\mathcal{M}(q_1^\theta) = \begin{bmatrix} p_1^\theta & q_1^\theta \\ p_0^\theta & q_0^\theta \end{bmatrix} \begin{bmatrix} r_0 r_1 + 1 & r_1 \\ r_0 & 1 \end{bmatrix},$$

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Fix $\theta \in (0,1) \setminus \mathbb{Q}$. Let $[r_j]_{j\in\mathbb{N}}$ be its unique continued fraction expansion, and

$$\mathcal{M}(q_1^\theta) = \begin{bmatrix} p_1^\theta & q_1^\theta \\ p_0^\theta & q_0^\theta \end{bmatrix} \begin{bmatrix} r_0 r_1 + 1 & r_1 \\ r_0 & 1 \end{bmatrix},$$

$$\mathcal{M}(q_n^\theta) = \begin{bmatrix} p_{n-1}^\theta & q_{n-1}^\theta \\ p_n^\theta & q_n^\theta \end{bmatrix}.$$

Set $\mathcal{AF}_{\theta,0} = \mathbb{C}$ and for all positive $n \in \mathbb{N}$,

$$\mathcal{AF}_{\theta,n} = \mathcal{M}(q_n^\theta) \oplus \mathcal{M}(q_{n-1}^\theta).$$

Let $\alpha_{\theta,n} : \mathcal{AF}_{\theta,n} \to \mathcal{AF}_{\theta,n+1}$ denote each unital *-morphism. The *Effros-Shen algebra* $\mathcal{AF}_\theta$ is given by

$$\mathcal{AF}_\theta = \varinjlim (\mathcal{AF}_{\theta,n}, \alpha_{\theta,n})_{n\in\mathbb{N}}.$$

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is continuous from $\theta \in (0,1) \setminus \mathbb{Q}$ with its topology as a subset of $\mathbb{R}$, to a distinguished class of quantum compact metric spaces metrized by the quantum propinquity Λ.

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which is explicitly given as the group:

$$\mathbb{Z}\left[\frac{1}{p}\right] = \left\{ \frac{z}{p^k} \in \mathbb{Q} : z \in \mathbb{Z}, k \in \mathbb{N} \right\}$$

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A *noncommutative solenoid* is a C^*-algebra of the form

$$C^*\left(\left[\frac{1}{p}\right] \times \left[\frac{1}{p}\right], \sigma\right),$$

where σ is a multiplier of the group $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$.

Theorem (Packer, Latrémolière 2016)

Noncommutative solenoids are limits for the quantum Gromov-Hausdorff propinquity of some $\{\mathcal{A}_{\theta_j}\}$!