Brown's Spectral Measure, and the Free Multiplicative Brownian Motion

West Coast Operator Algebras Seminar
Seattle University

Todd Kemp
UC San Diego

October 7, 2018

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Brownian Motion

Segal–Bargmann

Brown Measure Support

Dedication

This talk, and all my work, is dedicated to the memory of my father:

Robin Edward Kemp September 9, 1938 – August 4, 2018

who was a brilliant, hard-working, gentle, and humble man, and is the source of my strength, my intellect, and my success.

Giving Credit where Credit is Due

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- Biane, P.: *Segal-Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems*. J. Funct. Anal. 144, 1, 232-286 (1997)
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- Collins, Dahlqvist, K: *The Spectral Edge of Unitary Brownian Motion.* Probab. Theory Related Fields 170, no. 102, 49-93 (2018)
- Hall, K: *Brown Measure and the Free Multiplicative Brownian Motion.* arXiv:1810.00153

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Brown's Spectral Measure

Brown's Spectral Measure in Tracial von Neumann Algebras

If $(\mathcal{A}, \tau)$ is a W^*-probability space, then any normal operator $a \in \mathcal{A}$ has a spectral measure $\mu_a = \tau \circ E^a$. If A is a normal matrix, μ_A is its ESD. It is characterized (nicely) by the $*$-distribution of a:

$$\int_{\mathbb{C}} z^k \bar{z}^\ell \, \mu_a(dz d\bar{z}) = \tau(a^k a^{*\ell}).$$

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$$\int_{\mathbb{C}} z^k \bar{z}^\ell \, \mu_a(dz d\bar{z}) = \tau(a^k a^{*\ell}).$$

If a is not normal, there is no such measure. But there is a substitute: Brown's spectral measure. Let $L(a)$ denote the (log) Kadison–Fuglede determinant:

$$L(a) = \int_{\mathbb{R}} \log t \, \mu_{|a|}(dt) = \tau\left(\int_{\mathbb{R}} \log t \, E^{|a|}(dt) \right)$$

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(the last $=$ holds if $a^{-1}\in\mathcal{A}$).

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(the last $=$ holds if $a^{-1} \in \mathcal{A}$). Then $\lambda \mapsto L(a - \lambda)$ is subharmonic on $\mathbb{C}$, and

$$\mu_a = \frac{1}{2\pi} \nabla_\lambda^2 L(a - \lambda)$$

is a probability measure on $\mathbb{C}$. If A is *any* matrix, μ_A is its ESD.

The Circular Law

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Consider a circular operator z (mentioned yesterday in Brent Nelson's talk):

$$z = \frac{1}{\sqrt{2}}(x + iy) \qquad x, y \text{ freely independent semicirculars.}$$

It is not too difficult to compute from the definition that

$$\mu_z = \text{ uniform probability measure on } \overline{\mathbb{D}}.$$

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This goes hand in hand with the fact that z is the large-N limit (in $*$-distribution) of the Ginibre ensemble (all i.i.d. Gaussian entries), whose ESD converges to the uniform probability measure on $\overline{\mathbb{D}}$ (that's the Circular Law proved by Ginibre, Girko, Bai, Tao-Vu, . . .)

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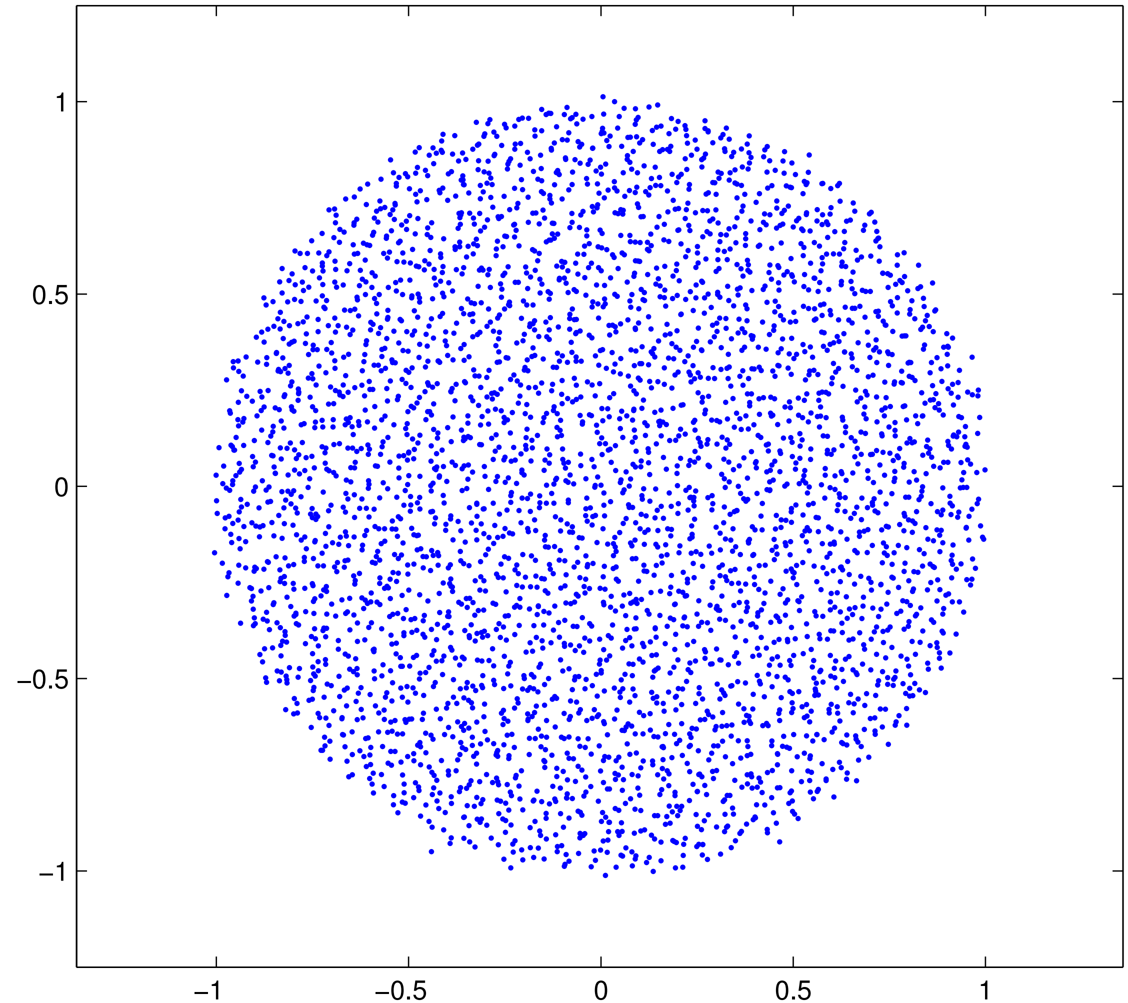

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This goes hand in hand with the fact that z is the large-N limit (in $*$-distribution) of the Ginibre ensemble (all i.i.d. Gaussian entries), whose ESD converges to the uniform probability measure on $\overline{\mathbb{D}}$ (that's the Circular Law proved by Ginibre, Girko, Bai, Tao-Vu, . . .)

However, the connection between limit ESD and Brown measure is actually very complicated.

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Properties of Brown Measure

The Brown measure has some nice properties analogous to the spectral measure, but not all:

- $\tau(a^k) = \int_{\mathbb{C}} z^k \, \mu_a(dz d\bar{z})$ and $\tau(a^{*k}) = \int_{\mathbb{C}} \bar{z}^k \, \mu_a(dz d\bar{z})$ but you *cannot max and match*.

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- $\tau(\log|a-\lambda|) = L(a-\lambda) = \int_{\mathbb{C}} \log|z-\lambda| \, \mu_a(dzd\bar{z})$ for large λ, and this characterizes μ_a.

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- supp $\mu_a \subseteq \mathrm{Spec}(a)$ (can be a strict subset).

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Let A^N be a sequence of matrices with a as limit in $*$-distribution. Since the Brown measure μ_{A^N} is the empirical spectral distribution of A^N, it is natural to expect that $\mathrm{ESD}(A^N) \to \mu_a$.

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Let A^N be a sequence of matrices with a as limit in $*$-distribution. Since the Brown measure μ_{A^N} is the empirical spectral distribution of A^N, it is natural to expect that $\mathrm{ESD}(A^N) \to \mu_a$. The $\log$ discontinuity often makes this exceedingly difficult to prove.

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Convergence of the Brown Measure

Let $\{a, a_n\}_{n\in\mathbb{N}}$ be a uniformly bounded set of operators in some W^*-probability spaces, with $a_n \to a$ in $*$-distribution. We would hope that $\mu_{a_n} \to \mu_a$. Without some very fine information about the spectral measure of $|a_n - \lambda|$ near the edge of $\mathrm{Spec}(a_n)$, the best that can be said in general is the following.

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Proposition. Suppose that $\mu_{a_n} \to \mu$ weakly for some probability measure μ on $\mathbb{C}$. Then

$$\int_{\mathbb{C}} \log|z-\lambda|\,\mu(dzd\bar{z}) \le \int_{\mathbb{C}} \log|z-\lambda|\,\mu_a(dzd\bar{z})$$

for all $\lambda \in \mathbb{C}$; and equality holds for sufficiently large λ.

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for all $\lambda \in \mathbb{C}$; and equality holds for sufficiently large λ.

Corollary. Let V_a be the unbounded connected component of $\mathbb{C} \setminus \operatorname{supp}\mu_a$. Then $\operatorname{supp}\mu \subseteq \mathbb{C} \setminus V_a$. (In particular, if $\operatorname{supp}\mu_a$ is simply-connected, then $\operatorname{supp}\mu \subseteq \operatorname{supp}\mu_a$.)

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The function $L(a-\lambda) = \int_{\mathbb{R}} \log t\, \mu_{|a|}(dt)$ is essentially impossible to compute with. But we can use regularity properties of the spectral resolution to approach it in a different way. Define

$$L^{\epsilon}(a) = \frac{1}{2}\tau(\log(a^*a+\epsilon)), \qquad \epsilon > 0.$$

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The function $\lambda \mapsto L^\epsilon(a-\lambda)$ is $C^\infty(\mathbb{C})$, and is subharmonic. Define

$$h_a^\epsilon(\lambda) = \frac{1}{2\pi}\nabla_\lambda^2 L_\epsilon(a-\lambda).$$

Then h_a^ϵ is a smooth probability density on $\mathbb{C}$, and

$$\mu_a(d\lambda) = \lim_{\epsilon\downarrow 0} h_a^\epsilon(\lambda)\, d\lambda.$$

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It is not difficult to explicitly calculate the density h_a^ϵ for fixed $\epsilon > 0$.

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The Density h_a^ϵ and the Spectrum of a

Lemma. Let $\lambda \in \mathbb{C}$, and denote $a_\lambda = a - \lambda$. Then

$$h_a^\epsilon(\lambda) = \frac{1}{\pi}\epsilon\tau\left((a_\lambda^* a_\lambda + \epsilon)^{-1}(a_\lambda a_\lambda^* + \epsilon)^{-1}\right).$$

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From here it is easy to see why supp $\mu_a \subseteq \mathrm{Spec}(a)$. If $\lambda \in \mathrm{Res}(a)$ so that $a_\lambda^{-1} \in \mathcal{A}$, we quickly estimate

$$\begin{aligned}&\left|\tau\left((a_\lambda^* a_\lambda + \epsilon)^{-1}(a_\lambda a_\lambda^* + \epsilon)^{-1}\right)\right| \\ \le &\left\|(a_\lambda^* a_\lambda + \epsilon)^{-1}(a_\lambda a_\lambda^* + \epsilon)^{-1}\right\|\end{aligned}$$

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&\le \|(a-\lambda)^{-1}\|^4.
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&\leq \|(a-\lambda)^{-1}\|^4.
\end{aligned}$$

This is locally uniformly bounded in λ; so taking $\epsilon \downarrow 0$, the factor of ϵ in $h_a^\epsilon(\lambda)$ kills the term; we find $\mu_a = 0$ in a neighborhood of λ.

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Invertibility in $L^p(\mathcal{A}, \tau)$

Recall that $L^p(\mathcal{A}, \tau)$ is the closure of $\mathcal{A}$ in the norm

$$\|a\|_p^p = \tau(|a|^p) = \tau\left((a^*a)^{p/2}\right).$$

(It can be realized as a set of densely-defined unbounded operators, acting on the same Hilbert space as $\mathcal{A}$). The non-commutative L^p-norms satisfy the same Hölder inequality as the classical ones.

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Recall that $L^p(\mathcal{A},\tau)$ is the closure of $\mathcal{A}$ in the norm

$$\|a\|_p^p = \tau(|a|^p) = \tau\Big((a^*a)^{p/2}\Big)\,.$$

(It can be realized as a set of densely-defined unbounded operators, acting on the same Hilbert space as $\mathcal{A}$). The non-commutative L^p-norms satisfy the same Hölder inequality as the classical ones.

It is perfectly possible for $a \in \mathcal{A}$ to be *invertible in* $L^p(\mathcal{A},\tau)$ without having a bounded inverse. That is: there can exist $b \in L^p(\mathcal{A},\tau) \setminus \mathcal{A}$ with $ab = ba = 1$ (viewed as an equation in $L^p(\mathcal{A},\tau)$).

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Invertibility in $L^p(\mathcal{A}, \tau)$

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The preceding proof (with very little change) shows that $h_a^\epsilon(\lambda) \to 0$ at any point λ where $a - \lambda$ is invertible in $L^4(\mathcal{A}, \tau)$.

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The preceding proof (with very little change) shows that $h_a^\epsilon(\lambda) \to 0$ at any point λ where $a - \lambda$ is invertible in $L^4(\mathcal{A},\tau)$.

Definition. The $L^p(\mathcal{A},\tau)$ *resolvent* $\mathrm{Res}_{p,\tau}(a)$ is the interior of the set of $\lambda \in \mathbb{C}$ for which $a - \lambda$ has an inverse in $L^p(\mathcal{A},\tau)$. The $L^p(\mathcal{A},\tau)$ *spectrum* $\mathrm{Spec}_{p,\tau}(a)$ is $\mathbb{C} \setminus \mathrm{Res}_{p,\tau}(a)$.

The $L^p(\mathcal{A}, \tau)$ Spectrum

From Hölder's inequality, we have the inclusions

$$\mathrm{Spec}_{p,\tau}(a) \subseteq \mathrm{Spec}_{q,\tau}(a) \subseteq \mathrm{Spec}(a)$$

for $1 \leq p \leq q < \infty$. Without including the closure in the definition, these inclusions can be strict; with the closure, my (wild) conjecture is that $\mathrm{Spec}_{1,\tau}(a) = \mathrm{Spec}(a)$ for all a.

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$$\frac{\pi}{\epsilon} h_a^\epsilon(\lambda) = \tau\big((a_\lambda^* a_\lambda + \epsilon)^{-1}(a_\lambda a_\lambda^* + \epsilon)^{-1}\big)\,.$$

If we naïvely set $\epsilon = 0$ on the right-hand-side, we get (heuristically)

$$\tau\left((a_\lambda^* a_\lambda)^{-1}(a_\lambda a_\lambda^*)^{-1})\right) = \tau\left((a_\lambda^*)^{-1}(a_\lambda)^{-2}(a_\lambda^*)^{-1}\right)$$

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Note, this is *not* equal to $\|a_\lambda^{-1}\|_4^4$ when a_λ is not normal.

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The $L^2_{2,\tau}$ Spectrum

Proposition. Let $a \in \mathcal{A}$, and suppose a^2 is invertible in $L^2(\mathcal{A}, \tau)$. Then for all $\epsilon > 0$,

$$\tau\big((a^*a + \epsilon)^{-1}(aa^* + \epsilon)^{-1}\big) \leq \|a^{-2}\|_2^2.$$

(The proof is trickier than you might think.)

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Definition. The $L^2_{2,\tau}$ *resolvent* of a, $\mathrm{Res}^2_{2,\tau}(a)$, is the interior of the set of $\lambda \in \mathbb{C}$ for which $(a-\lambda)^2$ is invertible in $L^2(\mathcal{A},\tau)$. The $L^2_{2,\tau}$ *spectrum* of a is $\mathrm{Spec}^2_{2,\tau}(a) = \mathbb{C} \setminus \mathrm{Res}^2_{2,\tau}(a)$.

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Theorem. supp $\mu_a \subseteq \mathrm{Spec}^2_{2,\tau}(a)$.

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Brown Measure Support

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Theorem. supp $\mu_a \subseteq \mathrm{Spec}^2_{2,\tau}(a)$.

Another wild conjecture: this is actually equality. (That depends on showing that, if a^2 is *not* invertible in $L^2(\mathcal{A},\tau)$, the above quantity blows up at rate $\Omega(1/\epsilon)$. This appears to be what happens in the case that a is normal, which would imply $\mathrm{Spec}^2_{2,\tau}(a) = \mathrm{Spec}_{4,\tau}(a)$ $= \mathrm{Spec}(a)$ in that case.)

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Brownian Motion on $\mathrm{U}(N)$, $\mathrm{GL}(N,\mathbb{C})$, and the Large-$N$ Limit

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Brown Measure Support

On any Riemannian manifold M, there's a Laplace operator Δ_M. And where there's a Laplacian, there's a Brownian motion: the Markov process $(B_t^x)_{t\geq 0}$ on M with generator $\frac{1}{2}\Delta_M$, started at $B_0^x = x \in M$.

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Let Γ be a (matrix) Lie group. Any inner product on $\mathrm{Lie}(\Gamma) = T_I\Gamma$ gives rise to a unique left-invariant Riemannian metric, and corresponding Laplacian Δ_Γ. On Γ we canonically start the Brownian motion $(B_t)_{t\geq 0}$ at $I \in \Gamma$.

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There is a beautiful relationship between the Brownian motion W_t on the Lie algebra $\mathrm{Lie}(\Gamma)$ and the Brownian motion B_t: the *rolling map*

$$dB_t = B_t \circ dW_t, \qquad \text{i.e.} \qquad B_t = I + \int_0^t B_t \circ dW_t.$$

Here $\circ$ denotes the Stratonovich stochastic integral. This can always be converted into an Itô integral; but the answer depends on the structure of the group Γ (and the chosen inner product).

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Brown Measure Support

Fix the *reverse normalized* Hilbert–Schmidt inner product on $\mathbb{M}_N(\mathbb{C})$ for all matrix Lie algebras:

$$\langle A, B\rangle = N\mathrm{Tr}(B^* A).$$

Let $X_t = X_t^N$ and $Y_t = Y_t^N$ be independent Hermitian Brownian motions of variance t/N.

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The Brownian motion on $\mathrm{Lie}(\mathrm{U}(N))$ is iX_t; the Brownian motion U_t on $\mathrm{U}(N)$ satisfies

$$\boxed{dU_t = iU_t\,dX_t - \tfrac{1}{2}U_t\,dt.}$$

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Brownian Motion on $\mathrm{U}(N)$ and $\mathrm{GL}(N,\mathbb{C})$

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The Brownian motion on $\mathrm{Lie}(\mathrm{GL}(N,\mathbb{C})) = \mathbb{M}_N(\mathbb{C})$ is $Z_t = 2^{-1/2}i(X_t + iY_t)$; the Brownian motion G_t on $\mathrm{GL}(N,\mathbb{C})$ satisfies

$$\boxed{dG_t = G_t\,dZ_t.}$$

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Brown Measure Support

Free Additive Brownian Motion

If $X_t = X_t^N$ is a Hermitian Brownian motion process, then at each time $t > 0$ it is a GUE_N with entries of variance t/N. Wigner's law then shows that the empirical spectral distribution of X_t^N converges to the semicircle law $\varsigma_t = \frac{1}{2\pi t}\sqrt{(4t - x^2)_+}\,dx$.

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A process $(x_t)_{t\geq 0}$ (in a W^*-probability space with trace τ) is a **free additive Brownian motion** if its increments are freely independent — $x_t - x_s$ is free from $\{x_r : r \leq s\}$ — and $x_t - x_s$ has the semicircular distribution ς_{t-s}, for all $t > s$.

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In 1991, Voiculescu showed that the Hermitian Brownian motion $(X_t^N)_{t\ge0}$ converges to $(x_t)_{t\ge0}$ in finite-dimensional non-commutative distributions:

$$\frac{1}{N}\mathrm{Tr}(P(X_{t_1},\ldots,X_{t_n})) \to \tau(P(x_{t_1},\ldots,x_{t_n})) \qquad \forall P.$$

Free Unitary and Free Multiplicative Brownian Motion

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Brown Measure Support

There is now a well-developed theory of free stochastic differential equations. Initially constructed in the free Fock space setting (by Kümmerer and Speicher in the early 1990s), it was used by Biane in 1997 to define "free versions" of U_t and G_t.

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Free Unitary and Free Multiplicative Brownian Motion

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Let x_t, y_t be freely independent free additive Brownian motions, and $z_t = 2^{-1/2}i(x_t + iy_t)$. The **free unitary Brownian motion** is the process started at $u_0 = 1$ defined by

$$\boxed{du_t = iu_t\,dx_t - \frac{1}{2}u_t\,dt.}$$

The **free multiplicative Brownian motion** is the process started at $g_0 = 1$ defined by

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It is natural to expect that these processes should be the large-N limits of the $\mathrm{U}(N)$ and $\mathrm{GL}(N, \mathbb{C})$ Brownian motions.

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Brown Measure Support

Free Unitary Brownian Motion

Theorem. [Biane, 1997] For all non-commutative (Laurent) polynomials P in n variables and times $t_1, \ldots, t_n \geq 0$,

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Free Unitary Brownian Motion

- Dedication
- Citations

Brown Measure

Brownian Motion
- BM on Lie Groups
- $\mathbb{U}$ & $\mathbb{GL}$
- Free$+$BM
- Free$\times$BM
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- Transforms
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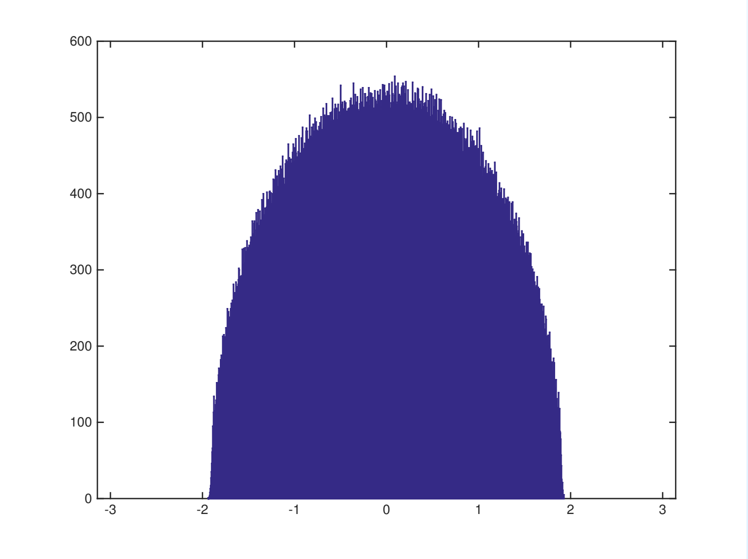

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Analytic Transforms Related to u_t

Biane's approach to understanding the measure ν_t was through its moment-generating function

$$\psi_t(z) = \int_{\mathbb{U}} \frac{uz}{1-uz}\,\nu_t(du) = \sum_{n\geq 1} m_n(\nu_t)\,z^n$$

(the second $=$ holds for $|z|<1$; the integral converges for $1/z \notin$ supp ν_t).

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(the second $=$ holds for $|z|<1$; the integral converges for $1/z \notin \operatorname{supp}\nu_t$). Then define

$$\chi_t(z) = \frac{\psi_t(z)}{1+\psi_t(z)}.$$

The function χ_t is injective on $\mathbb{D}$, and has a one-sided inverse f_t: $f_t(\chi_t(z)) = z$ for $z\in\mathbb{D}$ (but $\chi_t\circ f_t$ is only the identity on a certain region in $\mathbb{C}$; more on this later).

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Using the SDE for u_t and some clever complex analysis, Biane showed that

$$f_t(z) = z e^{\frac{t}{2}\frac{1+z}{1-z}}.$$

The Large-N Limit of G_t^N

In 1997 Biane conjectured a similar large-N limit should hold for the Brownian motion on $\mathrm{GL}(N,\mathbb{C})$, but the ideas of his U_t^N proof (spectral theorem, representation theory of $\mathrm{U}(N)$) did not translate well to the a.s. non-normal process G_t^N.

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Theorem. [K, 2014 (2016)] For all non-commutative Laurent polynomials P in $2n$ variables, and times $t_1, \ldots, t_n \geq 0$,

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The proof required several new ingredients: a detailed understanding of the Laplacian on $\mathrm{GL}(N,\mathbb{C})$, and concentration of measure for trace polynomials. Putting these together with an iteration scheme from the SDE, together with requisite covariance estimates, yielded the proof.

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This is convergence of the (multi-time) $*$-distribution, of a *non-normal* matrix process. What about the eigenvalues?

The Eigenvalues of Brownian Motion $\mathrm{GL}(N, \mathbb{C})$

Because U_t^N and u_t are normal, their $*$-distributions encode their ESDs, so the bulk eigenvalue behavior is fully understood.

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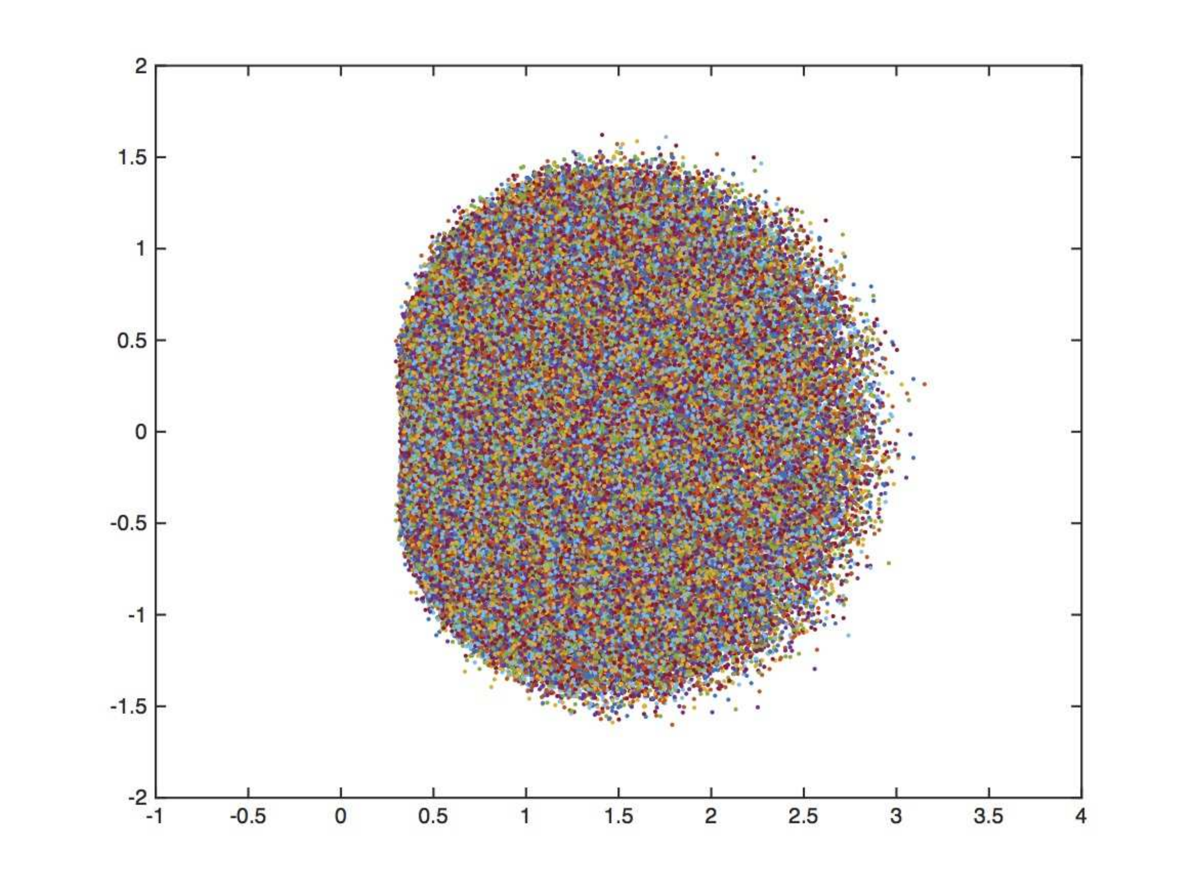

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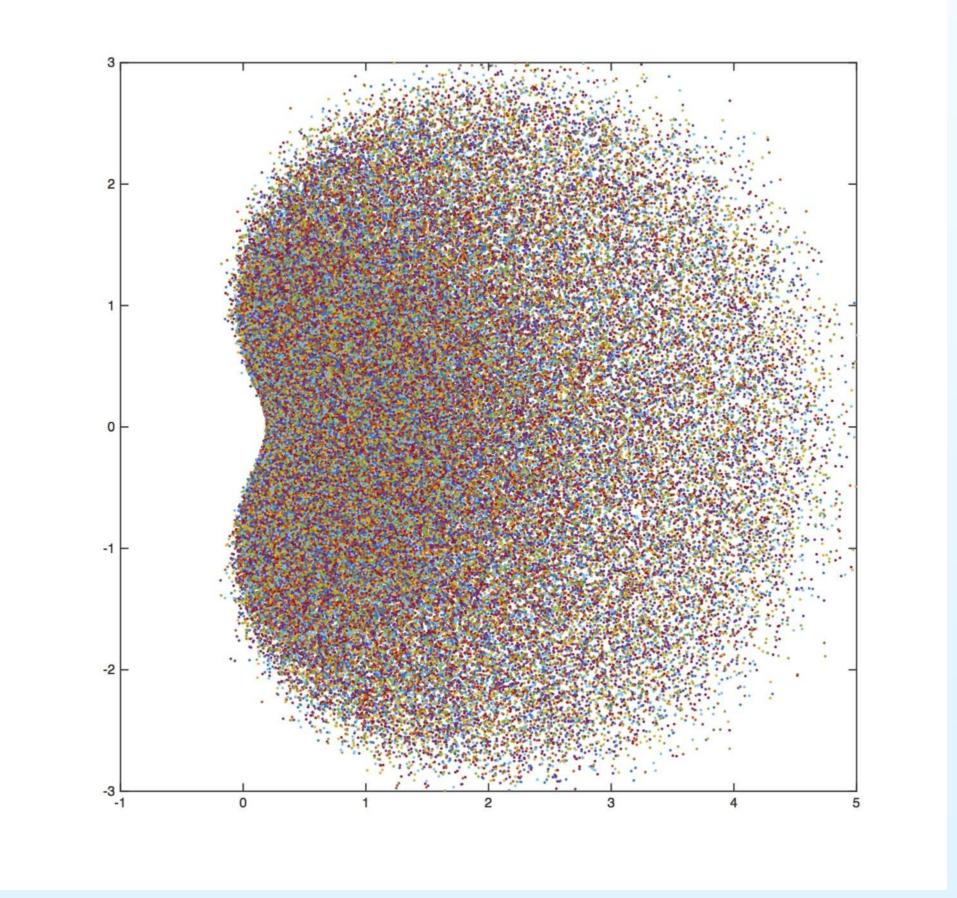

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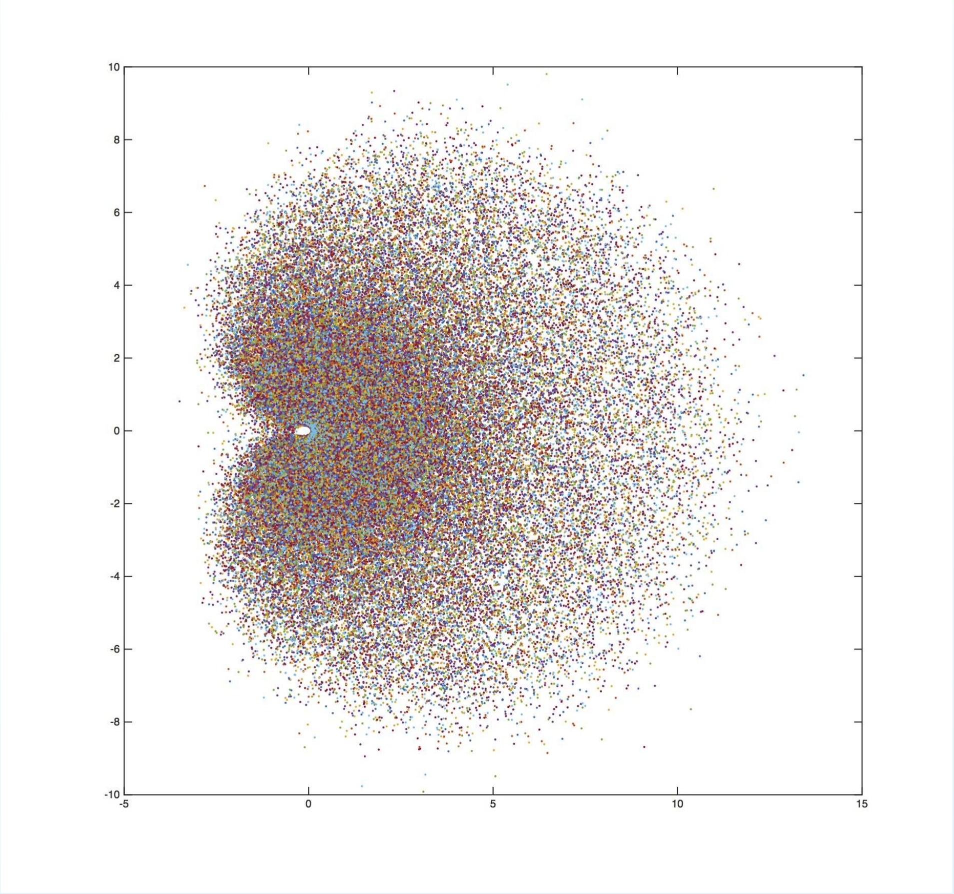

$t = 4$

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The Segal–Bargmann Transform

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The Unitary Segal–Bargmann Transform

The **Segal–Bargmann (Hall) Transform** is a map from functions on $\mathrm{U}(N)$ to holomorphic functions on $\mathrm{GL}(N,\mathbb{C})$. It is defined by the analytic continuation of the action of the heat operator:

$$\mathbf{B}_t^N f = \left(e^{\frac{t}{2}\Delta_{\mathrm{U}(N)}} f\right)_{\mathbb{C}}.$$

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Writing out what this integral formula means in probabilistic terms, here is a nice way to express it: let F already be a holomorphic function on $\mathrm{GL}(N),\mathbb{C})$, and let $f = F|_{\mathrm{U}(N)}$. Let U_t and G_t be independent Brownian motions on $\mathrm{U}(N)$ and $\mathrm{GL}(N,\mathbb{C})$. Then

$$(\mathbf{B}_t f)(G_t) = \mathbb{E}[F(G_t U_t)|G_t].$$

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$$(\mathbf{B}_t f)(G_t) = \mathbb{E}[F(G_t U_t)|G_t].$$

This extends beyond f that already possess an analytic continuation; it defines an *isometric isomorphism*

$$\mathbf{B}_t^N : L^2(\mathrm{U}(N), U_t) \to \mathcal{H}L^2(\mathrm{GL}(N,\mathbb{C}), G_t).$$

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The Free Unitary Segal–Bargmann Transform

In 1997, Biane introduced a free version of the Unitary SBT, which can be described in similar terms: acting on, say, polynomials f in a single variable, $\mathscr{G}_t f$ is defined by

$$(\mathscr{G}_t f)(g_t) = \tau[f(g_t u_t)|g_t].$$

He conjectured that $\mathscr{G}_t$ is the large-N limit of $\mathbf{B}_t^N$ in an appropriate sense; this was proven by Driver, Hall, and me in 2013. (It was for this work that we invented trace polynomial concentration.)

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Biane proved directly (and it follows from the large-N limit) that $\mathscr{G}_t$ extends to an isometric isomorphism

$$\mathscr{G}_t \colon L^2(\mathbb{U}, \nu_t) \to \mathscr{A}_t$$

where $\mathscr{A}_t$ is a certain reproducing-kernel Hilbert space of holomorphic functions. The norm on $\mathscr{A}_t$ is given by

$$\|F\|_{\mathscr{A}_t}^2 = \tau(|F(g_t)|^2) = \tau(F(g_t)^* F(g_t)) = \|F(g_t)\|_2^2.$$

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The Range of the Free Segal–Bargmann Transform

The functions $F \in \mathscr{A}_t$ are not all entire functions. They are holomorphic on a bounded region Σ_t

$$\Sigma_t = \mathbb{C} \setminus \overline{\chi_t(\mathbb{C} \setminus \text{supp}\, \nu_t)}$$

where (recall) χ_t is the (right-)inverse of $f_t(z) = ze^{\frac{t}{2}\frac{1+z}{1-z}}$.

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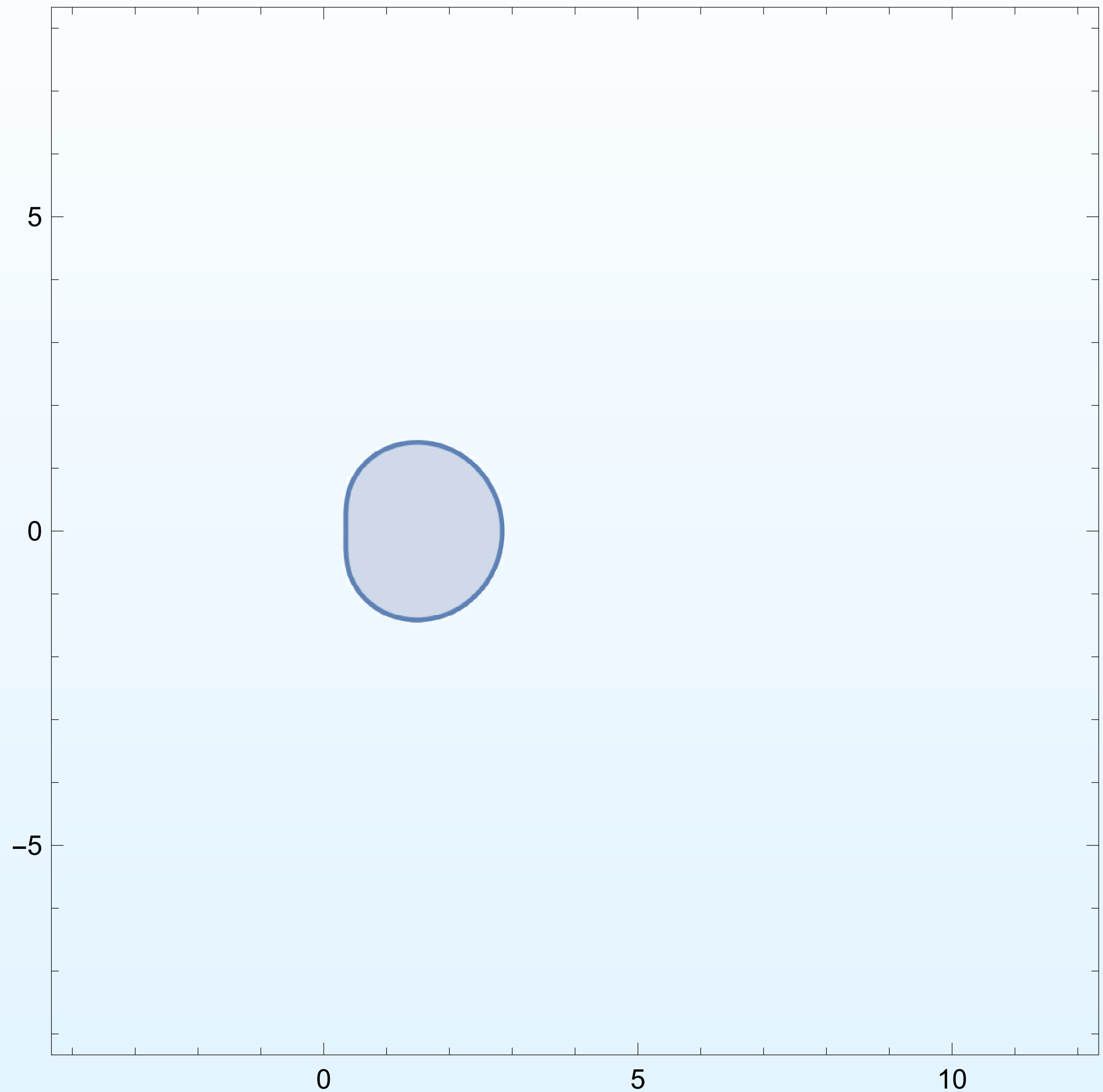

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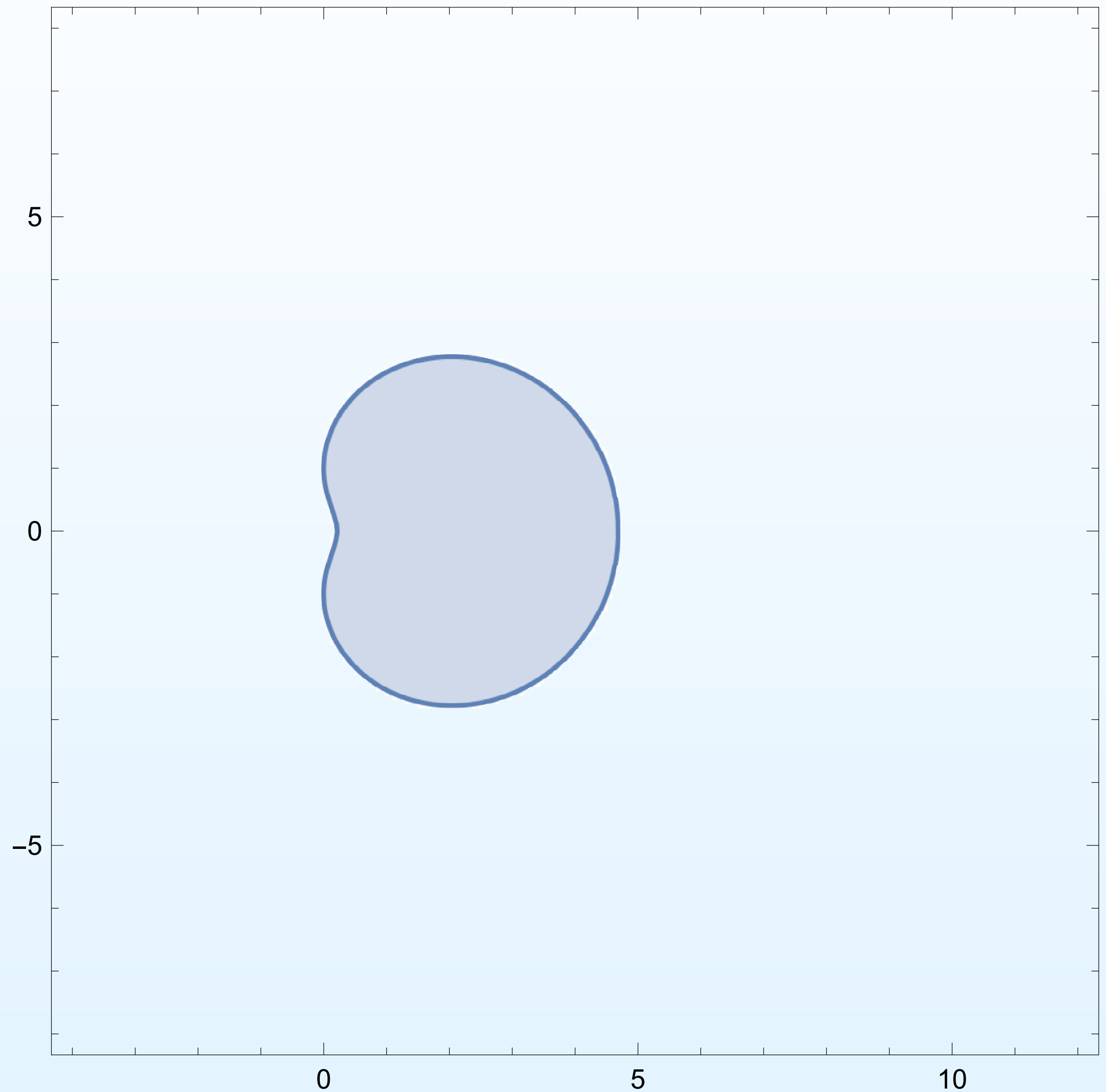

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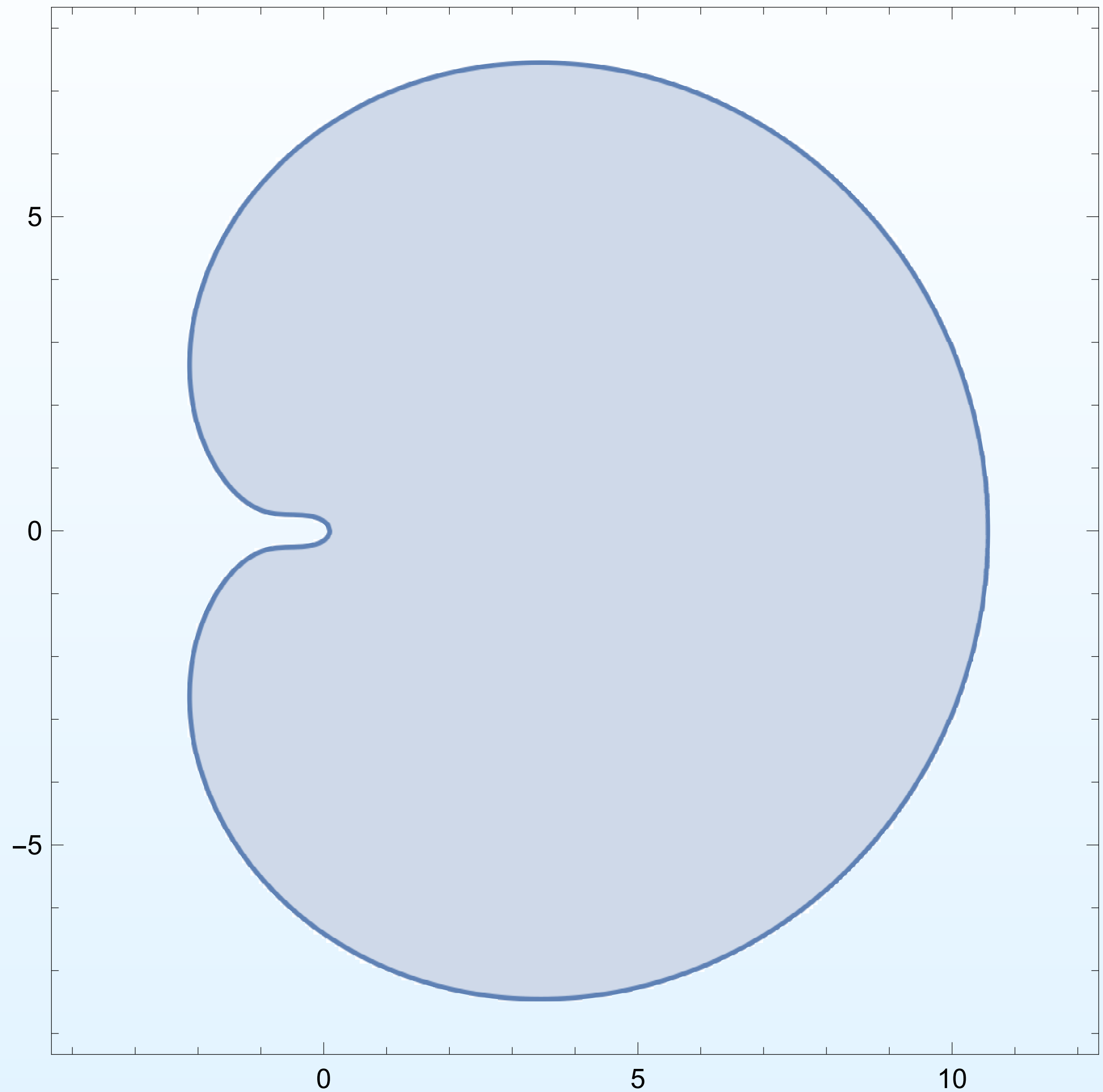

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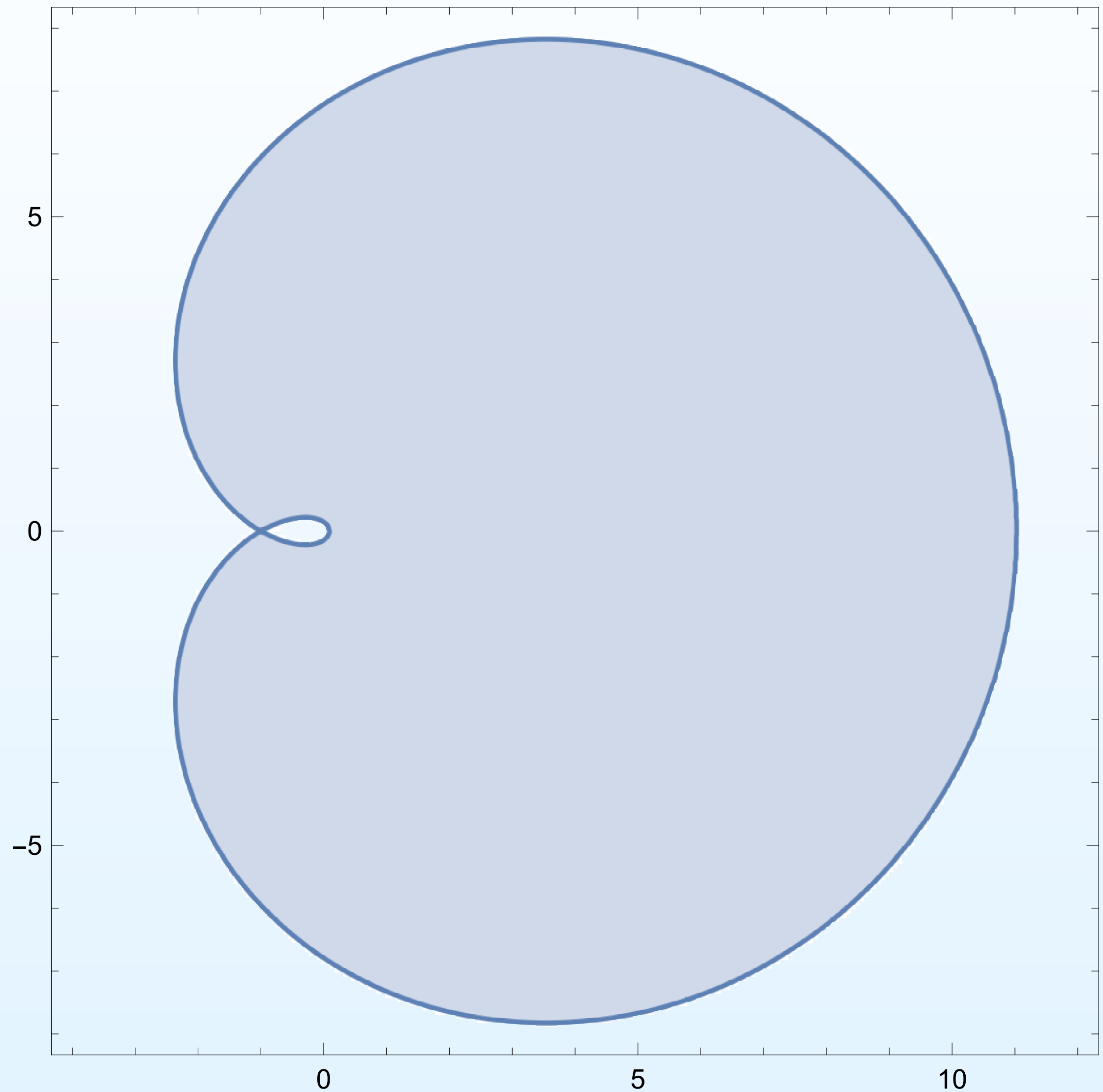

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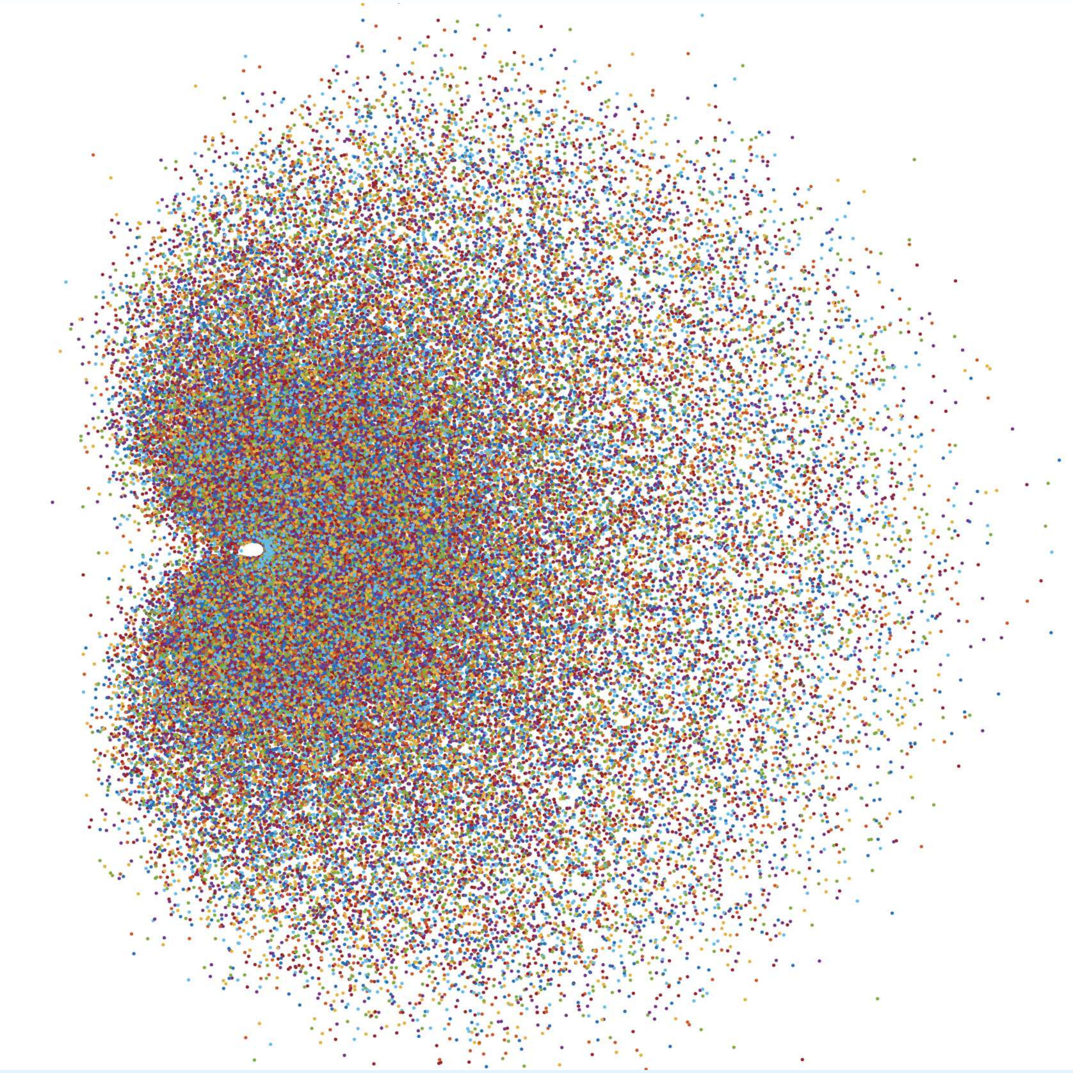

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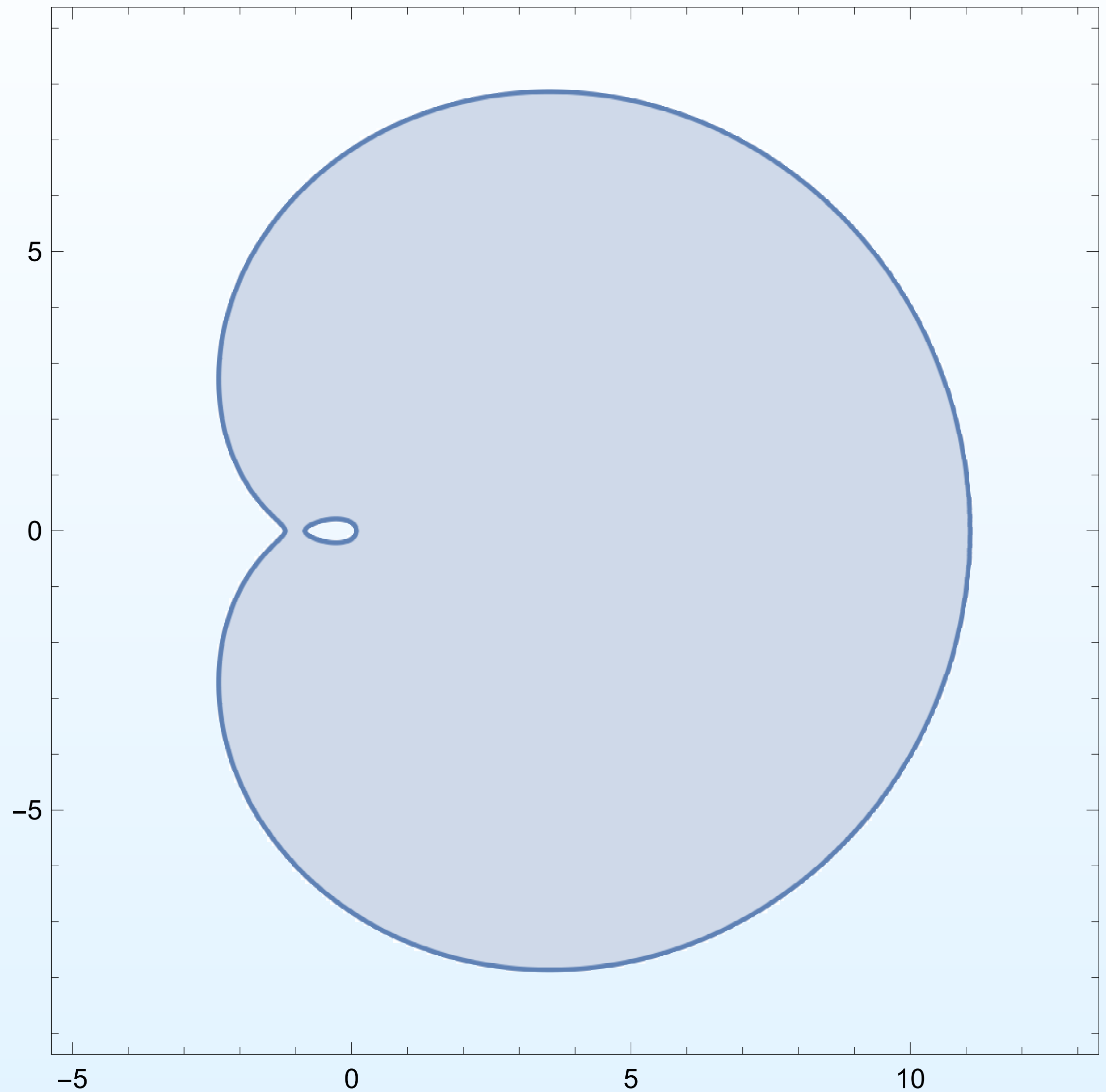

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Theorem. (Hall, K, 2018)

$$\mathrm{supp}\mu_{g_t} \subseteq \overline{\Sigma}_t.$$

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Essentially, $\lambda \in \mathrm{Res}^2_{2,\tau}(g_t)$ iff $(g_t - \lambda)^2$ is invertible in $L^2(\tau)$, i.e.

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Using PDE techniques, we can compute that

$$\mathscr{G}_t^{-1}((z-\lambda)^{-1}) = \frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda) - u}.$$

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$$\mathscr{G}_t : \frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda)-u} \mapsto \frac{1}{z-\lambda}.$$

Since $\frac{1}{(z-\lambda)^2} = \frac{d}{d\lambda}\frac{1}{z-\lambda}$, using regularity properties of $\mathscr{G}_t$ we have

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$$\alpha_t^\lambda(u) = \frac{d}{d\lambda}\left(\frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda)-u}\right).$$

The question is: for which λ is $\alpha_t^\lambda \in L^2(\mathbb{U}, \nu_t)$? I.e.

$$\int_{\mathbb{U}} |\alpha_t^\lambda(u)|^2 \, \nu_t(du) < \infty.$$

The Support of The Brown Measure of g_t

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$$\mathscr{G}_t : \frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda)-u} \mapsto \frac{1}{z-\lambda}.$$

Since $\frac{1}{(z-\lambda)^2} = \frac{d}{d\lambda}\frac{1}{z-\lambda}$, using regularity properties of $\mathscr{G}_t$ we have

$$\alpha_t^\lambda(u) = \frac{d}{d\lambda}\left(\frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda)-u}\right).$$

The question is: for which λ is $\alpha_t^\lambda \in L^2(\mathbb{U},\nu_t)$? I.e.

$$\int_{\mathbb{U}} |\alpha_t^\lambda(u)|^2\, \nu_t(du) < \infty.$$

The answer is: precisely when $f_t(\lambda) \notin \operatorname{supp}\nu_t$. I.e.

$$\mathrm{Res}^2_{2,\tau}(g_t) = f_t^{-1}(\mathbb{C}\setminus \operatorname{supp}\nu_t) = \chi_t(\mathbb{C}\setminus\operatorname{supp}\nu_t).$$

$\square$

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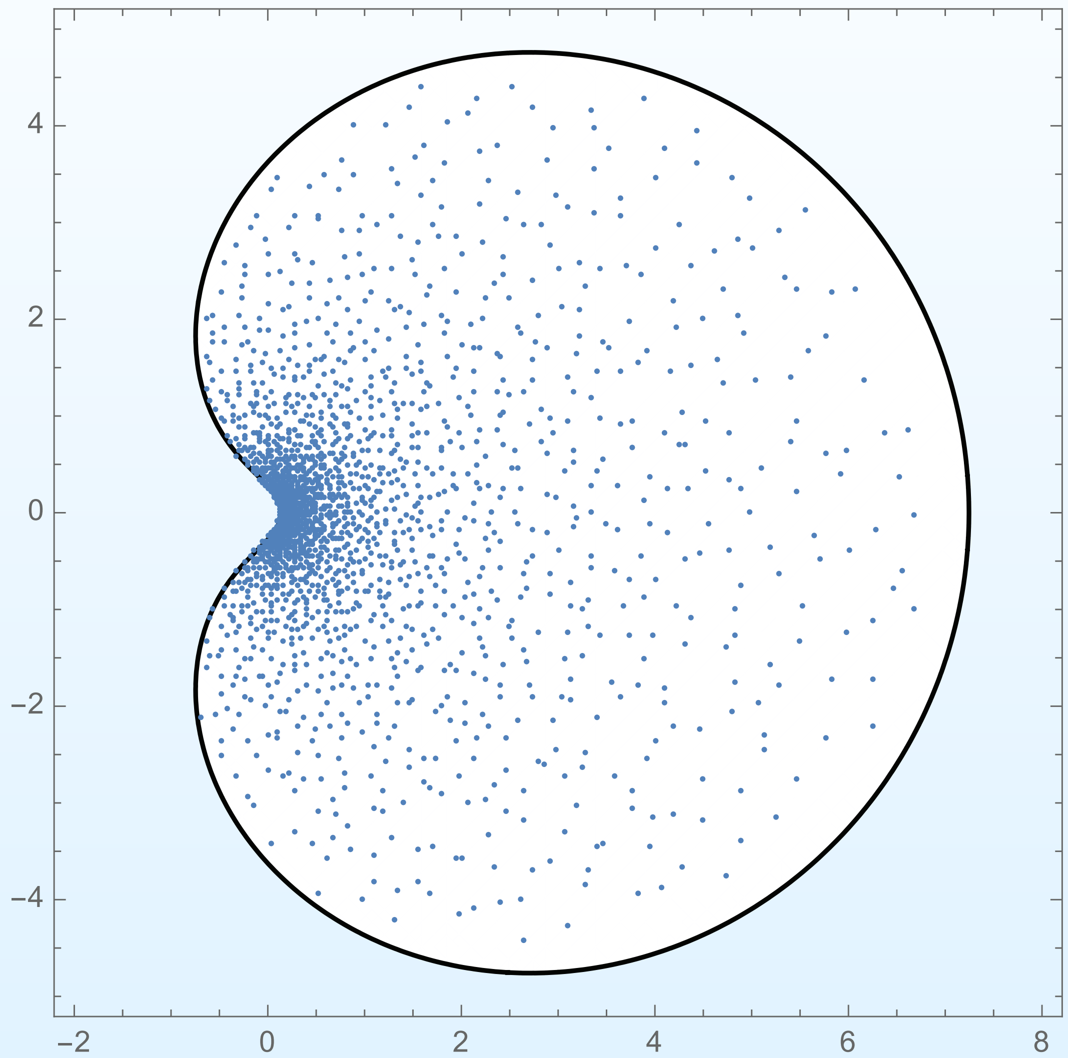
Here is a simulation of eigenvalues of $G_t^{(N)}$ for $N = 2000$, together with the boundary of Σ_t, at $t = 3$ (produced in Mathematica).



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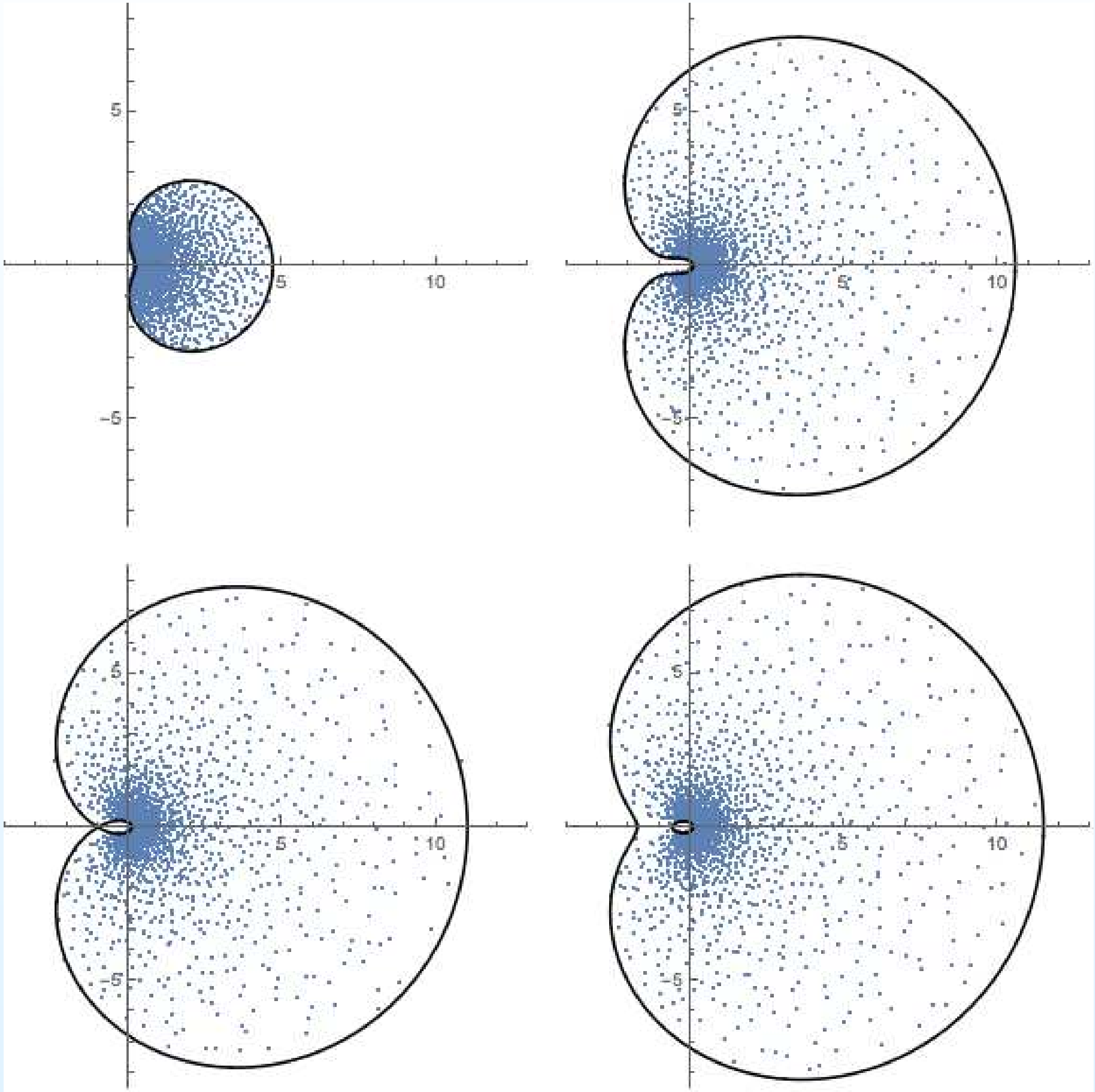
$N = 2000, t = 2, 3.9, 4, 4.1.$



Computing the Brown Measure

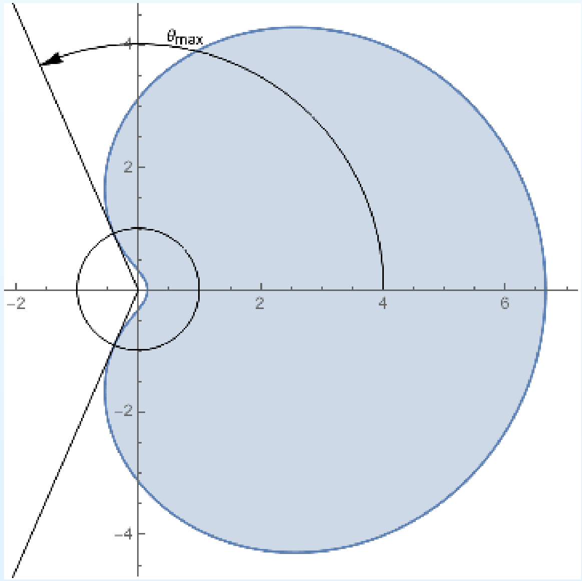
Very recently, jointly with Driver and Hall, we have been able to push further and actually compute the Brown measure.

Computing the Brown Measure

Very recently, jointly with Driver and Hall, we have been able to push further and actually compute the Brown measure. To describe it, we need an auxiliary implicit function $\varrho = \varrho(t,\theta)$, determined by

$$\frac{1-\varrho\cos\theta}{\sqrt{1-\varrho^2}}\log\left(\frac{2-\varrho^2+2\sqrt{1-\varrho^2}}{\varrho^2}\right) = t.$$

This defines a real analytic function for $|\theta| < \theta_{\max}(t) = \cos^{-1}(1-t/2)\wedge\pi$, which is precisely the argument range of Σ_t:



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Theorem. (Driver, Hall, K, 2018)
$\operatorname{supp}\mu_{g_t} = \overline{\Sigma}_t.$

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The Brown Measure of g_t

Theorem. (Driver, Hall, K, 2018)
$\operatorname{supp}\mu_{g_t} = \overline{\Sigma}_t$. Moreover, μ_{g_t} has a continuous density on $\overline{\Sigma}_t$, that is real analytic and strictly positive on Σ_t.

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Theorem. (Driver, Hall, K, 2018)
$\operatorname{supp}\mu_{g_t} = \overline{\Sigma}_t$. Moreover, μ_{g_t} has a continuous density on $\overline{\Sigma}_t$, that is real analytic and strictly positive on Σ_t. The density has the form

$$d\mu_{g_t} = \frac{1}{r^2} w_t(e^{i\theta}) \mathbb{1}_{\overline{\Sigma}_t}\, r\,dr\,d\theta$$

for the real analytic function $w_t \colon \mathbb{U} \to \mathbb{R}_+$ given by

$$w_t(e^{i\theta}) = \frac{1}{4\pi}\left(\frac{2}{t} + \frac{\partial}{\partial\theta}\frac{\varrho(t,\theta)\sin\theta}{1-\varrho(t,\theta)\cos\theta}\right).$$

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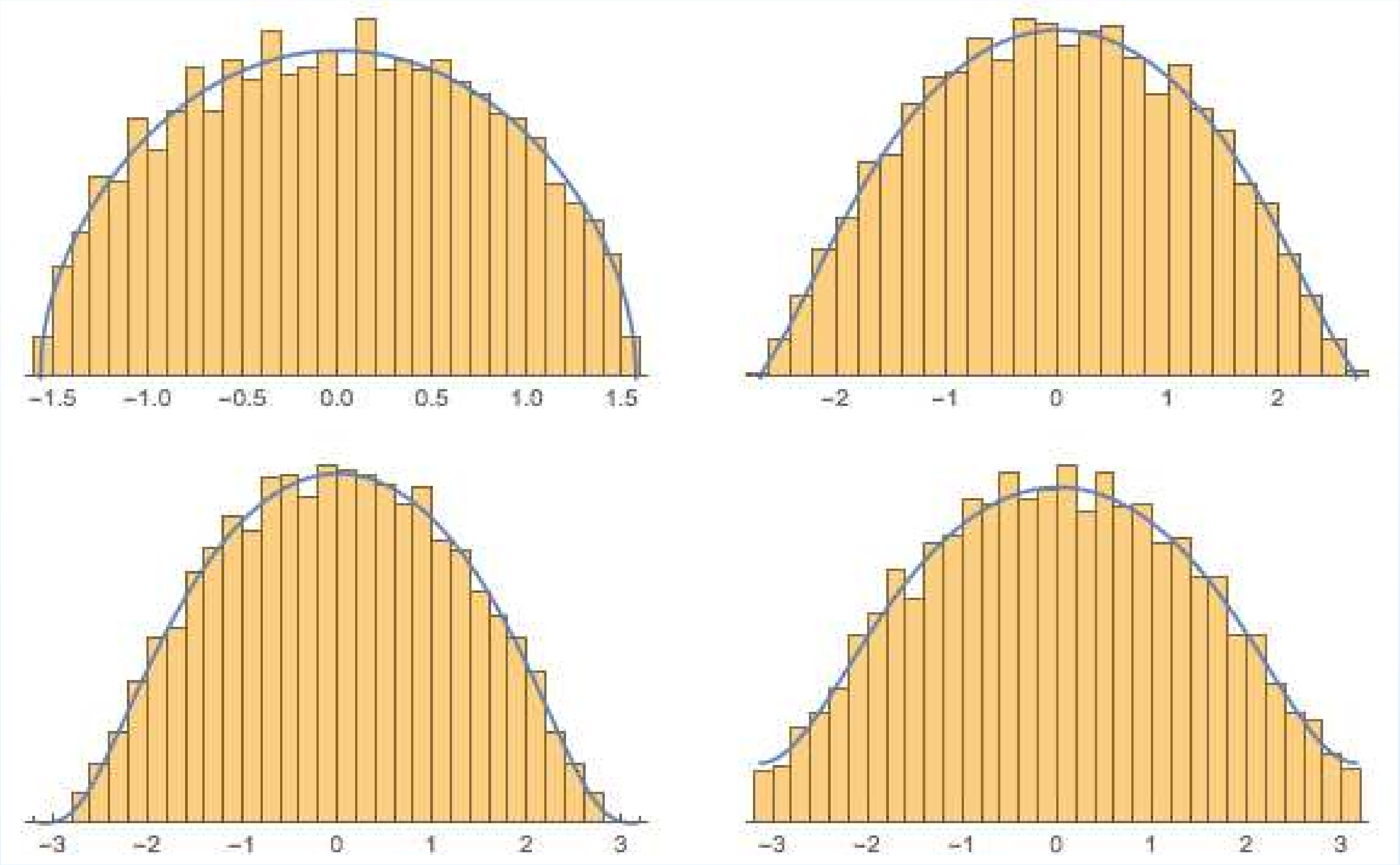
The techniques needed to prove this theorem are wholly disjoint from the concepts discussed in this talk so far; they rely primarily on PDE methods. You'll have to wait to see those ideas until the next meeting (Oberwolfach, or Montreal).

Histogram of Eigenvalue Arguments

Here are histograms of complex arguments of eigenvalues of G_t^N, together with the argument density of μ_{g_t}, for $N = 2000$ and $t = 2, 3.8, 4, 5$.



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Remaining Questions

- Explore relations between the $L^p(\tau)$-spectra, in general. They are probably all equal to the spectrum for g_t.

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- Explore relations between the $L^p(\tau)$-spectra, in general. They are probably all equal to the spectrum for g_t.
- Prove that the ESD of G_t^N actually converges to μ_{g_t}. (What we can now say definitively is that the limit ESD is supported in $\overline{\Sigma}_t$.)

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- Prove that the ESD of G_t^N actually converges to μ_{g_t}. (What we can now say definitively is that the limit ESD is supported in $\overline{\Sigma}_t$.)
- There is a two-parameter family of invariant diffusions on $\mathrm{GL}(N, \mathbb{C})$ that includes U_t^N and G_t^N, all of which have large-N limits described by free SDEs. How much of all this extends to the whole family? (Our preprint already covers the support in the two-parameter setting; the density is yet unknown.)

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I'll let you know what more I know next time we meet. ○