A Proof of Analytic Subordination for Free Additive Convolution using Monotone Independence

David Jekel

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1 Overview

This talk is going to be more expository, although I'll mention a few of my own results at the end if there's time. I'm hoping that you'll be able to understand most of it if you're not a specialist in non-commutative probability.

For the specialists, I want to mention that everything I'm about to say about free, Boolean, and monotone independence will generalize to the operator-valued setting with the same proofs. But to keep the exposition simple, I'll focus on the scalar-valued case.

This talk is going to have mainly two parts. In the first half, I'll give a survey of different types of independence — classical, free, boolean, monotone, and anti-monotone. In the second half, I'll explain the proof of analytic subordination advertised in the title.

2 Non-commutative Independences

2.1 Non-commutative Probability Spaces

For our purposes, a *non-commutative probability space* consists of a unital $*$-algebra $\mathcal{A}$ and a state $E : \mathcal{A} \to \mathbb{C}$. We think of the elements of $\mathcal{A}$ as bounded random variables and E as the expectation.

This framework includes classical probability theory. Indeed, in classical probability theory, we take $\mathcal{A}$ to be $L^\infty(\Omega, P)$ and E to be the classical expectation. $L^\infty(\Omega, P)$ is explicitly realized as an algebra of operators on the Hilbert space $\mathcal{H} = L^2(\Omega, P)$, since each L^∞ function acts on $L^2(\Omega, P)$ by multiplication. The expectation is then given by the vector state $E[T] = \langle \xi, T\xi \rangle$, where ξ is the function 1 in $L^2(\Omega, P)$.

More generally, given a non-commutative probability space $(\mathcal{A}, E)$, we can use the GNS construction to realize $\mathcal{A}$ as an algebra of operators on a Hilbert space $\mathcal{H}$ with a distinguished vector ξ such that $E[T] = \langle \xi, T\xi \rangle$. Hence, $\mathcal{A}$ can be completed to a C^* or W^* algebra if desired.

2.2 Philosophy of Independence

Classical and non-commutative probability theory deal with various notions of independence. Independence can be viewed as a rule for determining the joint law of two (or more) random variables based on their individual laws. In classical probability theory, if two bounded random variables X and Y are independent, then that uniquely determines $E[f(X,Y)]$ for all polynomials f. Equivalently, it allows us to compute arbitrary mixed moments of X and Y; that is, we can compute the expectation of any string on the alphabet $\{X, Y\}$, e.g.

$$E[XYXXY] = E[X^3]E[Y^2].$$

More generally, if two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of bounded random variables are independent, then we can compute the expectation of any string of letters from $\mathcal{A}_1$ and $\mathcal{A}_2$ based on $E|_{\mathcal{A}_1}$ and $E|_{\mathcal{A}_2}$.

This leads to the following working definition of the concept of independence: *A type of independence is a universal rule for computing mixed moments for two (or more) given algebras $\mathcal{A}_j$ in terms of $E|_{\mathcal{A}_j}$.*

2.3 Elements of a Type of Probability Theory

For all the types of independence that we will discuss, we'll have the following the tools / results. At the board, I will present this list in a chart, explaining the abstract version and the classical version simultaneously, then the free, then the boolean, then the monotone, then the anti-monotone.

1. **Definition:** a rule for computing joint moments of elements of two algebras.

2. **Product construction:** given Hilbert spaces with distinguished unit vectors $(\mathcal{H}_1, \xi_1)$ and $(\mathcal{H}_2, \xi_2)$, we can define a product space $(\mathcal{H}, \xi)$ and $*$-homomorphisms $\rho_j : B(\mathcal{H}_j) \to B(\mathcal{H})$ such that $B(\mathcal{H}_1)$ and $B(\mathcal{H}_2)$ are independent with respect to $\langle \xi, \cdot \xi \rangle$ and $\langle \xi, \rho_j(T)\xi \rangle = \langle \xi_j, T\xi_j \rangle$. This leads to a product construction for algebras.

3. **Convolution:** The convolution of two laws μ and ν is the law of $X + Y$, where $X \sim \mu$ and $Y \sim \nu$.

4. **Analytic transforms:** Analytic functions associated to a law μ which aid in the computation of convolutions.

5. **Central Limit Theorem:** If μ has mean zero and variance 1, then the N-fold convolution of μ, rescaled by $N^{-1/2}$, converges to some universal limiting law.

6. **Combinatorial theory:** There are combinatorial formulas to systematically compute the expectation of a string with letters from $\mathcal{A}_1$ and $\mathcal{A}_2$. These are also related to the analytic transforms and the construction of product spaces.

For the sake of time, I'll only mention the combinatorial aspects in passing and not actually state the results. I will not give complete proofs, but I will give some details in the monotone case since it is the least familiar and the most necessary for the subordination proof I'll present later.

2.4 Classical Independence

1. **Definition:** $\mathcal{A}_1$ and $\mathcal{A}_2$ commute and $E[a_1 a_2] = E[a_1]E[a_2]$.
2. **Product construction:** Define $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\xi = \xi_1 \otimes \xi_2$. The inclusions $B(\mathcal{H}_j) \to B(\mathcal{H})$ are given by tensoring with the identity.
3. **Convolution:** The classical convolution $\mu * \nu$.
4. **Analytic transforms:** The Fourier transform $\widehat{\mu}$ satisfies $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$.
5. **Central Limit Theorem:** Convergence to the standard normal $(2\pi)^{-1/2} e^{-x^2/2}\, dx$.
6. **Combinatorial theory:** There are cumulants defined using the partitions of $[n]$.

2.5 Free Independence

For background, see [Voi86] [Voi91], [Spe94].

1. **Definition:** If $a_1 \dots a_n$ is an alternating string of letters from $\mathcal{A}_1$ and $\mathcal{A}_2$ and $E[a_j] = 0$, then $E[a_1 \dots a_n] = 0$.
2. **Product construction:** Let $\mathcal{K}_j$ be the orthogonal complement of ξ_j in $\mathcal{H}_j$. Let
$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq i_2 \neq \dots \neq i_n} \mathcal{K}_{i_1} \otimes \dots \otimes \mathcal{K}_{i_n}.$$
For each $j \neq i_1$, $\rho_j(T)$ acts on the subspace
$$\mathcal{K}_{i_1} \otimes \dots \otimes \mathcal{K}_{i_n} \oplus \mathcal{K}_j \otimes \mathcal{K}_{i_1} \otimes \dots \otimes \mathcal{K}_{i_n} \cong (\mathbb{C} \oplus \mathcal{K}_j) \otimes \mathcal{K}_{i_1} \otimes \dots \otimes \mathcal{K}_{i_n} \cong \mathcal{H}_j \otimes \mathcal{K}_{i_1} \otimes \dots \otimes \mathcal{K}_{i_n}$$
by applying T to the first tensorand.
3. **Convolution:** The free convolution $\mu \boxplus \nu$.
4. **Analytic transforms:** Define the Cauchy-Stieltjes transform $G_\mu(z) = \int (z-x)^{-1}\, d\mu(x)$. This is defined on $\mathbb{C} \setminus \operatorname{supp}(\mu)$ and behaves like $1/z$ near ∞. The R-transform is given by $1/z + R_\mu(z) = G_\mu^{-1}(z)$ where defined (including a neighborhood of 0 when μ is compactly supported). We have $R_{\mu \boxplus \nu} = R_\mu + R_\nu$.
5. **Central Limit Theorem:** Convergence to the standard semicircular $(2\pi)^{-1}\sqrt{4 - x^2}\chi_{[-2,2]}(x)\, dx$.
6. **Combinatorial theory:** There are cumulants defined using the noncrossing or planar partitions of $[n]$.

2.6 Boolean Independence

For background, see [SW97].

1. **Definition:** $\mathcal{A}_1$ and $\mathcal{A}_2$ don't necessarily include the unit in the larger algebra $\mathcal{A}$, but they have internal units. If $a_1 \dots a_n$ is an alternating string of letters from $\mathcal{A}_1$ and $\mathcal{A}_2$, then $E[a_1 \dots a_n] = E[a_1] \dots E[a_n]$.

2. **Product construction:** Let
$$\mathcal{H} = \mathbb{C}\xi \oplus \mathcal{K}_1 \oplus \mathcal{K}_2.$$
We define $\rho_j(T)$ to act by T on $\mathbb{C}\xi \oplus \mathcal{K}_j \cong \mathcal{H}_j$ and to act by zero on the orthogonal complement. These inclusions are *non-unital*. Random variables X and Y are said to be independent if $\mathbb{C}[X]_0$ and $\mathbb{C}[Y]_0$ are independent, where $\mathbb{C}[x]_0$ denotes the polynomials with no constant term.

3. **Convolution:** The Boolean convolution $\mu \uplus \nu$.

4. **Analytic transforms:** The B-transform is given by $B_\mu(z) = 1/G_\mu(1/z) - 1/z$. We have $B_{\mu \uplus \nu} = B_\mu + B_\nu$.

5. **Central Limit Theorem:** Convergence to the standard Bernoulli $(1/2)\delta_{-1} + (1/2)\delta_1$.

6. **Combinatorial theory:** There are cumulants defined using the interval partitions of $[n]$.

2.7 Monotone and Anti-monotone Independence

For background, see [Mur97], [Mur00], [Mur01], [Has10a], [Has10b], [HS11].

1. **Definition:** $\mathcal{A}_1$ and $\mathcal{A}_2$ don't include unit in the larger algebra $\mathcal{A}$, but they have internal units. If $a_1 \dots a_n$ is a string of letters from $\mathcal{A}_1$ and $\mathcal{A}_2$, and if $a_j \in \mathcal{A}_2$ but the adjacent terms are in $\mathcal{A}_1$, then $E[a_1 \dots a_n] = E[a_1 \dots a_{j-1} E[a_j] a_{j+1} \dots a_n]$.

2. **Product construction:** Let
$$\mathcal{H} = \mathbb{C}\xi \oplus \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_2 \otimes \mathcal{K}_1 = \mathbb{C}\xi \oplus \bigoplus_{i_1 > i_2 > \dots > i_n} \mathcal{K}_{i_1} \otimes \dots \otimes \mathcal{K}_{i_n}.$$
$\rho_1(T)$ acts by T on $\mathbb{C}\xi \oplus \mathcal{K}_1$ and by zero on the orthogonal complement. Viewing $\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_2$, we define $\rho_2(T) = T \otimes \mathrm{id}$.

3. **Convolution:** The monotone convolution $\mu \rhd \nu$.

4. **Analytic transforms:** The F-transform is given by $F_\mu(z) = 1/G_\mu(z)$. We have $F_{\mu \rhd \nu} = F_\mu \circ F_\nu$.

5. **Central Limit Theorem:** Convergence to the standard arcsine law $1/\pi\sqrt{2 - x^2} \cdot \chi_{(-\sqrt{2}, \sqrt{2})}(x)\, dx$.

6. **Combinatorial theory:** There are cumulants defined using ordered non-crossing partitions of $[n]$, or using non-crossing partitions and multiplying by certain coefficients.

Unlike the other types, monotone independence is *not* commutative (with respect to the ordering of the algebras). Thus, monotone convolution is not commutative. For anti-monotone independence, we reverse the order of the algebras. We write $\triangleleft$ for the anti-monotone convolution operation.

2.8 Remarks on General Independences

The types of independence described above are the only ones which yield a product operation on probability spaces that is functorial and associative. This is a theorem of Muraki 2003 [Mur03], building on work of Speicher 1997 [Spe97] and Ben Ghorbal and Schürmann 2002 [BS02]. However, there are other functorial ways of joining N algebras if you don't require associativity; see for instance [Liu18, §3].

3 Some Details about Monotone Independence

We will now give some details about monotone independence in preparation for the proof we present later.

3.1 The Monotone Product Space

Lemma 1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with unit vectors ξ_1 and ξ_2. Let $\mathcal{H}$ be the monotone product space and $\rho_j : B(\mathcal{H}_j) \to B(\mathcal{H})$ be the inclusion map described above. Then $\rho_1(B(\mathcal{H}_1))$ and $\rho_2(B(\mathcal{H}_2))$ are monotone independent in $B(\mathcal{H})$ with respect to ξ.*

Proof. Let $\mathcal{A}_1 = \rho_1(B(\mathcal{H}_1))$ and $\mathcal{A}_2 = \rho_2(B(\mathcal{H}_2))$. We have to show that for a string of elements from $\mathcal{A}_1$ and $\mathcal{A}_2$, if $a_j \in \mathcal{A}_2$ and the terms next to it are in $\mathcal{A}_1$, then we can replace a_j by $E[a_j]$ without changing the expectation of the string. In fact, it suffices to show that $a_{j-1}a_ja_{j+1} = a_{j-1}E[a_j]a_{j+1}$.

Thus, it suffices to prove that $\rho_1(x)\rho_2(y)\rho_1(z) = \rho_1(x)E[y]\rho_1(z)$. By replacing y with $y - E[y]$, we can assume without loss of generality that $E[y] = 0$. Recall that the image of $\rho_1(z)$ is contained in $\mathbb{C}\xi \oplus \mathcal{K}_1$. Since $E[y] = 0$, we know that $\rho_2(y)$ maps $\mathbb{C}\xi$ into $\mathcal{K}_2$ and maps $\mathcal{K}_1$ into $\mathcal{K}_2 \otimes \mathcal{K}_1$. But $\rho_1(x)$ kills $\mathcal{K}_2 \oplus \mathcal{K}_2 \otimes \mathcal{K}_1$. Therefore, $\rho_1(x)\rho_2(y)\rho_1(z) = 0$. □

3.2 Monotone Convolution and Composition

Lemma 2. *We have $F_{\mu \triangleright \nu} = F_\mu \circ F_\nu$ for compactly supported measures μ and ν.*

Proof. Let $X \sim \mu$ and $Y \sim \nu$ be monotone independent. Let's denote

$$\tilde{G}_\mu(z) = G_\mu(z^{-1}) = E[(z^{-1} - X)^{-1}] = E[(1 - zX)^{-1} z].$$

Note that if inv is the involution $z \mapsto z^{-1}$, then

$$\tilde{G}_\mu = \operatorname{inv} \circ F_\mu \circ \operatorname{inv},$$

so it suffices to show that $\tilde{G}_{\mu \rhd \nu} = \tilde{G}_\mu \circ \tilde{G}_\nu$, and by analytic continuation, it suffices to prove this in a neighborhood of 0. Note that

$$\begin{aligned} \tilde{G}_{\mu \rhd \nu}(z) &= E[(1 - zX - zY)^{-1} z] \\ &= E[(1 - (1 - zY)^{-1} zX)^{-1} (1 - zY)^{-1} z] \\ &= E\left[\sum_{k=0}^{\infty} [(1 - zY)^{-1} zX]^k (1 - zY)^{-1} z\right]. \end{aligned}$$

Now $(1 - zY)^{-1}$ is 1 plus something in the closure of $\mathbb{C}[Y]_0$ and it is sandwiched between two occurences of $X \in \mathbb{C}[X]_0$. By monotone independence, we can replace $(1 - zY)^{-1} - 1$ by its expectation. Of course, 1 is already equal to its expectation. So overall we can replace each occurence of $(1 - zY)^{-1} z$ by its expectation which is $\tilde{G}_\nu(z)$. Therefore,

$$\tilde{G}_{\mu \rhd \nu}(z) = E\left[\sum_{k=0}^{\infty} [\tilde{G}_\nu(z) X]^k \tilde{G}_\nu(z)\right] = \tilde{G}_\mu \circ \tilde{G}_\nu(z). \qquad \square$$

4 Subordination

The following theorem relates free and monotone convolution.

Theorem 3. *Let μ and ν be compactly supported measures. Then there exists a compactly supported measure ν' such that $\mu \boxplus \nu = \mu \rhd \nu'$.*

Corollary 4. *$F_{\mu \boxplus \nu} = F_\mu \circ F_{\nu'}$ and hence $G_{\mu \boxplus \nu} = G_\mu \circ F_{\nu'}$, where $F_{\nu'} : \mathbb{H} \to \mathbb{H}$ analytic.*

This was first proved in Voiculescu's first free entropy paper [Voi93] by complex-analytic methods under some regularity assumptions on the Cauchy transforms. Voiculescu later gave another proof based on the properties of resolvents with respect to the non-commutative differentiation. Biane [Bia98] gave a proof by constructing the subordination function combinatorially in a neighborhood of ∞ and using properties of conditional expectation to show it extends to the whole upper half-plane. Belinschi, Mai, Speicher 2013 gave an analytic proof for the operator-valued case using the Earle-Hamilton theorem [BMS13].

This approach is due to Lenczewski 2007 [Len07, §7], and was extended to the multivariable case by Nica [Nic09, Remark 4.11] and the operator-valued case by Liu [Liu18, Proposition 7.2]. The measure ν' is denoted ... and is called the *subordination or s-free convolution* of μ and ν.

Proof of Theorem. The idea of the proof is to express the free product Hilbert space as a monotone product Hilbert space. Let's realize the laws μ and ν by operators X_1 on $(\mathcal{H}_1, \xi_1)$ and X_2 on $(\mathcal{H}_2, \xi_2)$. Let $\mathcal{H}_1 = \mathbb{C}\xi \oplus \mathcal{K}_1$, $\mathcal{H}_2 = \mathbb{C}\xi \oplus \mathcal{K}_2$. Let $\mathcal{H}$ be the free product Hilbert space

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{j_1 \neq \cdots \neq j_n} \mathcal{K}_{j_1} \otimes \cdots \otimes \mathcal{K}_{j_n}.$$

Define

$$\begin{aligned}\tilde{\mathcal{K}}_2 &= \bigoplus_{n \geq 1} \bigoplus_{\substack{j_1 \neq \cdots \neq j_n \\ j_n = 2}} \mathcal{K}_{j_1} \otimes \cdots \otimes \mathcal{K}_{j_n} \\ &= \mathcal{K}_2 \oplus (\mathcal{K}_1 \otimes \mathcal{K}_2) \oplus (\mathcal{K}_2 \otimes \mathcal{K}_1 \otimes \mathcal{K}_2) \oplus \ldots\end{aligned}$$

and $\tilde{\mathcal{H}}_2 = \mathbb{C}\xi \oplus \tilde{\mathcal{K}}_2$. Then we have

$$\mathcal{H} = \mathbb{C}\xi \oplus \mathcal{K}_1 \oplus \tilde{\mathcal{K}}_2 \oplus (\tilde{\mathcal{K}}_2 \otimes \mathcal{K}_1),$$

which is the monotone product of $(\mathcal{H}_1, \xi)$ and $(\tilde{\mathcal{H}}_2, \xi)$.

Now let $\rho_1 : B(\mathcal{H}_1) \to B(\mathcal{H})$ and $\rho_2 : B(\mathcal{H}_2) \to B(\mathcal{H})$ be the free product inclusions. Let $\tilde{\rho}_1 : B(\mathcal{H}_1) \to B(\mathcal{H})$ and $\tilde{\rho}_2 : B(\tilde{\mathcal{H}}_2) \to B(\mathcal{H})$ be the monotone product inclusions. Note that $\mu \boxplus \nu$ is the law of $\rho_1(X_1) + \rho_2(X_2)$. We want to construct a Z such that

$$\rho_1(X_1) + \rho_2(X_2) = \tilde{\rho}_1(X_1) + \tilde{\rho}_2(Z).$$

This will complete the proof with ν' being the law of Z. To construct Z, we consider the behavior of $\rho_1(X_1)$ and $\rho_2(X_2)$ separately.

First, consider $\rho_1(X_1)$. We view $\mathcal{H}_1 = \mathbb{C}\xi \oplus \mathcal{K}_1$ as a subspace of $\mathcal{H}$. This space is $\rho_1(X_1)$-invariant and hence

$$\rho_1(X_1) = P_{\mathcal{H}_1} \rho_1(X_1) P_{\mathcal{H}_1} + (1 - P_{\mathcal{H}_1}) \rho_1(X_1)(1 - P_{\mathcal{H}_1}).$$

The first term

$$P_{\mathcal{H}_1} \rho_1(X_1) P_{\mathcal{H}_1} = \tilde{\rho}_1(X_1).$$

The second term $(1 - P_{\mathcal{H}_1})\rho_1(X_1)(1 - P_{\mathcal{H}_1})$ maps $\tilde{\mathcal{K}}_2$ into itself, and the way it acts on $\tilde{\mathcal{K}}_2 \otimes \mathcal{K}_1$ is the same as the way it acts on $\tilde{\mathcal{K}}_2$ since $\rho_1(X_1)$ only acts on the leftmost factors of each tensor product. Let X_1' be this operator on $\tilde{\mathcal{K}}_2$ and extend it by zero to an operator on $\tilde{\mathcal{H}}_2$. Then $(1 - P_{\mathcal{H}_1})\rho_1(X_1)(1 - P_{\mathcal{H}_1}) = \tilde{\rho}_2(X_1')$.

Second, consider $\rho_2(X_2)$. We claim that this is $\tilde{\rho}_2$ of something else. Note that $\mathbb{C}\xi \oplus \tilde{\mathcal{K}}_2$ is invariant under $\rho_2(X_2)$. Indeed, applying $\rho_2(X_2)$ can only add or delete a factor $\mathcal{K}_2$ at the beginning of a string, so it can never relate the strings that end with 2 with the strings that end with 1. Let X_2' be $\rho_2(X_2)$ restricted to $\tilde{\mathcal{H}}_2$. Then $\rho_2(X_2) = \tilde{\rho}_2(X_2')$. Therefore, overall,

$$\rho_1(X_1) + \rho_2(X_2) = \tilde{\rho}_1(X_1) + \tilde{\rho}_2(X_1') + \tilde{\rho}_2(X_2') = \tilde{\rho}_1(X_1) + \tilde{\rho}_2(X_1' + X_2')$$

as desired. $\square$

This proof had numerous advantages. It doesn't require any analytic work. It is basically a computation with Hilbert spaces, and it goes through verbatim to the operator-valued setting. It shows automatically that the subordination function is an F-transform and one can deduce from the proof a combinatorial formula for the moments of ν'. One can also use the monotone independence to show that

$$E[(z - X + Y)^{-1}|X] = (F_{\nu'}(z) - X)^{-1}$$

(compare Biane's treatment). As a point of propaganda, this shows that monotone (and s-free) independence is "useful" for free probability, and it makes sense to study non-commutative probability as a whole rather than only free probability.

There is an analogous result, also noticed by Lenczewski concerning anti-monotone and boolean independence. This can be proved by the same method (although in this case the analytic proof is also easy).

Theorem 5. *For compactly supported measures μ and ν, there exists ν' such that $\mu \triangleleft \nu = \mu \uplus \nu'$.*

5 Advertisements

In the remaining time, I want to give a quick advertisement for my current work on these topics. On my UCLA department website, I have a long set of notes about operator-valued non-commutative probability (free, Boolean, and monotone) that includes these results. This includes an explanation of fully matricial or non-commutative functions (the operator-valued version of complex analysis). There is also a unified explanation of convolution semigroups for the three types of operator-valued independence (which were studied earlier in various papers).

The punch line is that for a convolution semigroup μ_t, the *cumulants* are given by the *moments* of $t\sigma$ for some "operator-valued measure" σ. The F-transforms evolve according to the equation

$$\partial_t F_t(z) = \begin{cases} -D_z F_t(z) \cdot G_\sigma(F_t(z)), & \text{free case} \\ -G_\sigma(z), & \text{boolean case} \\ -D_z F_t(z) \cdot G_\sigma(z), & \text{(anti-)monotone case} \\ -G_\sigma(F_t(z)), & \text{(anti-)monotone case.} \end{cases}$$

The correspondence between μ_t and G_σ defines the Bercovici-Pata bijection for these case.

Now such semigroups correspond to processes with independent and stationary increments (e.g. Brownian motion). More generally, for a process with independent and non-stationary increments, we have a similar differential equation except that σ is replaced by a time-dependent σ_t.

$$\partial_t F_t(z) = \begin{cases} -D_z F_t(z) \cdot G_{\sigma_t}(F_t(z)), & \text{free case} \\ -G_{\sigma_t}(z), & \text{boolean case} \\ -D_z F_t(z) \cdot G_{\sigma_t}(z), & \text{monotone case} \\ -G_{\sigma_t}(F_t(z)), & \text{anti-monotone case.} \end{cases}$$

In the operator-valued case, due to issues with differentiation in Banach spaces, the generalized law σ_t depends on t in some distributional sense rather than being a function $t \mapsto \sigma_t$. Here we assume that the support of μ_t is bounded and that the variance of μ_t is Lipschitz in t. But in the end, we get a Bercovici-Pata-like bijection for processes with independent increments.

These results are not in my notes, but the monotone case is in the paper [Jek17], where I generalize the observation of Schleißinger [Sch17] that subordination chains of functions on the upper half-plane (Loewner chains) correspond precisely to such processes with monotone independent increments.

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